Pivotal Persuasion*

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Abstract

A sender seeks to persuade a group of heterogenous voters to adopt an action. We analyze the sender's information-design problem when the collective decision is made through a majority vote and voting for the action is personally costly. We show that the sender can exploit the heterogeneity in voting costs by privately communicating with the voters. Under the optimal information structure, voters with lower costs are more likely to vote for the sender's preferred action when it is the wrong choice than those with higher costs. The sender's preferred action is, therefore, adopted with a higher probability when private communication is allowed than when it is not. Nevertheless, the sender's preferred action cannot be adopted with probability one if no voter, as a dictator, is willing to vote for it without being persuaded.

Keywords: Bayesian Persuasion, Information Design, Private Persuasion, Strategic Voting.

JEL Classification Codes: D72, D83

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1 Introduction

This paper considers an information-design problem of a sender who seeks to persuade a group of voters to support an action when voting for this action is costly for the voters. Examples of this type of group persuasion include an interest group lobbying legislators for a bill that is unpopular among their constituents, a CEO trying to convince a board of directors to support a controversial project, or a candidate urging their supporters to turn out to vote.

In our model, a group of N voters must decide between two alternatives: a and b. Action b is the default choice; action a is chosen only if it receives at least $K < N$ votes. There are two states, labeled A and B. Action a is the right choice in state A. Specifically, all voters prefer a in state A but are indifferent between a and b in state B. While action a is a weakly better choice, a voter must incur an extra cost to vote for a. The cost must be paid regardless of the outcome of the vote. A legislator who votes for an unpopular bill is less likely to be reelected even when the bill does not pass.[1] A director who supports a CEO's pet project may suffer a loss in reputation. In some countries, supporters of opposition candidates may be subject to intimidation at the poll. We assume that the cost for voting for a is heterogenous across voters, such that, *ceteris paribus*, some voters are more inclined to vote for a. A sender tries to lobby the voters to vote for a. We model the lobbying process as a persuasion game (Kamenica and Gentzkow 2011). First, the sender commits to a signal structure that maps each state to a probability distribution of private signal profiles. The chosen signal structure is publicly observed by all voters. Then, a vector of private signals is drawn according to the signal structure. Finally, voting takes place after each voter observes her own private signal. Following the literature, we focus on the sender's favorite equilibrium.

If the voters knew the state, our model would become a standard threshold public-good game (Palfrey and Rosenthal 1984). Voters would vote against a in state B, and there would be multiple equilibria in state A. In the best equilibrium for the sender, exactly K voters would vote for a, while the rest would free ride. When the voters do not know the state, the sender can increase the probability of a being chosen by designing a signal structure judiciously. On one hand, the sender wants the signals to be noisy, such that the voters would vote for a in state B with some

[1] On August 10, 1964, two senators, Ernest Gruening and Wayne Morse, voted against a joint congressional resolution allowing the president "to take all necessary steps, including the use of armed force, to assist any member or protocol state of the Southeast Asia Collective Defense Treaty requesting assistance in defense of its freedom." In 1968 election, they were both defeated, which was believed to be due to their votes. See http://thehill.com/opinion/campaign/365818-congress-has-forgotten-how-put-principle-above-politics

probability. On the other hand, the signals must be sufficiently accurate, such that the voters are confident when the state is A. Because of the incentives to free ride, the sender must convince a voter not only that voting for a is the right thing to do (i.e., the state is A) but also that her vote is needed.

As a benchmark, we consider the case where the sender is restricted to public communication. In our model, this amounts to requiring voters' signals to be perfectly correlated. In this case, the best the sender can do is to target the voter with the K-th lowest voting cost. If this voter is convinced by a public signal to vote for a, then the $K-1$ voters with lower voting costs will also be convinced to vote for a. Hence, when restricted to public communication, our model is equivalent to one where the voter with the K-th lowest voting cost is a dictator.[2] The solution to the latter problem is well known since Kamenica and Gentzkow (2011). The optimal signal should obfuscate the states so that, conditional on voting for a, the voter with the K-th lowest voting cost is indifferent between voting for a and voting for b.

In general, there is no reason why the sender must conduct all communication in public. In fact, interest groups often lobby legislators in private. In our model, the sender can use private communication to his advantage. Under the optimal information structure, every vote for a is pivotal in equilibrium. That is, conditional on receiving a signal that will lead her to vote for a, a voter knows that exactly $K-1$ other voters also receive signals that will lead them to vote for a. But, because multiple groups of K voters may vote for a in equilibrium, a voter who knows that her vote is pivotal for a will not be able to infer which group of $K-1$ other voters are voting for a. This allows the sender to fully exploit the heterogeneity in voting costs. Under the optimal information structure, a voter with a lower voting cost will have a strictly higher probability of voting for a in state B than a voter with a higher voting cost. In comparison, when communication is public, the K voters with the lowest voting costs will all vote for a with the same probability in state B. Thus, the sender can get a chosen with a higher probability when private communication is allowed than when it is not. However, the sender's ability to manipulate the outcome is limited. In particular, the sender cannot get a adopted with probability one if no voter, as a dictator, is willing to vote for a without any extra information. Our results sharply contrast with the case where voters' payoffs depend only on the collective decision and not on their individual votes. In such a model, a voter is indifferent about how she votes whenever her vote is not pivotal, and the sender can exploit this property to get a selected with a probability arbitrarily close to one, so long as the voters strictly

[2] We call a voter as a dictator if the collective decision coincides with her vote regardless of other voters' choices.

prefer a in state A. From the perspective of institutional design, the lesson is that a voting body is less susceptible to manipulation when voters must pay a price for their individual votes.

Throughout the paper, we assume that the sender can commit to any information structure. The sender can first commission a series of reports or experiments, and then reveal, to different experts, different subsets of the reports and experimental outcomes. Each expert then produces a report on the basis of the information that he observes and reveals it to a different voter. Different experts may have different idiosyncratic preferences over the actions. By choosing which experts to hire and what information each has access to, the sender can influence the correlation between the expert reports. We assume that the set of reports and the pool of experts are large enough to produce any information structure. This assumption enables us to highlight key intuitions. In reality, the set of feasible information structures may be limited. From the perspective of the voters, our exercise can be viewed as a "robust approach" to evaluate the manipulability of majority voting rules.[3]

Related Literature. Our work belongs to the growing literature about information design and Bayes correlated equilibria (BCE) developed by Bergemann and Morris (2013, 2016a,b, 2017) and Taneva (2014). As in the aforementioned papers, we use a linear programming approach to characterize the sender's optimal implementable information structures. Our contribution is to analyze the benefit of private persuasion in a strategic voting environment. When multiple voters receive private signals and then interact strategically, voters form private beliefs, and their higher-order beliefs are also strategically relevant. Mathevet, Perego, and Taneva (2017) propose an epistemic approach to information design in a multiple-agent setting.

Our paper is also related to the Bayesian persuasion literature, which uses a belief-based approach to analyze information-design problems. In a one-sender-one-receiver model, Kamenica and Gentzkow (2011) show that the sender's problem is simply to split the receiver's belief about the state subject to the standard Bayes' rule (or Bayes plausibility condition).[4] Alonso and Câmara (2016) consider public persuasion in a Bayesian persuasion game between one sender and multiple voters. They show that, when there are more than two payoff-dependent states, the optimal public signal may give rise to multiple distinct winning coalitions that adopt the sender's preferred action. Schnakenberg (2015) obtains a similar result in a cheap-talk model. In our model, because there are only two payoff-dependent states, and voters have common preference in each state, the

[3] A similar approach was adopted by Goldstein and Huang (2016) and Inostroza and Pavan (2018) who study information design in coordination games and evaluate different forms of stress test.

[4] See also Brocas and Carrillo (2007) and Rayo and Segal (2010).

optimal public signal always targets the K voters who are the easiest to persuade, and it is strictly dominated by private signals. Wang (2015) considers private persuasion with independent and identically distributed (i.i.d.) signals and shows that public persuasion performs as well as private persuasion. We allow the sender to use any correlated signals and highlight its contribution to the advantage of private persuasion. Bardhi and Guo (2018) investigate a persuasion game under unanimous rule when voters' preferences are correlated.

The comparison between private and public persuasion has also been studied in non-voting environments. Inostroza and Pavan (2018) study information design in a global game where a policy maker designs a stress test to minimize the chance of regime change. They also show that private persuasion dominates public one as in our paper. In their model, the advantage of private persuasion does not come from the possibility of tailoring the information disclosure to the agents' prior beliefs. Instead, private persuasion dominates as it makes it harder for the agents to coordinate on a successful attack. In a general setting, Arieli and Babichenko (2016) derive a sufficient and necessary condition under which public and conditionally independent signals are optimal. Also see Farrell and Gibbons (1989) and Goltsman and Pavlov (2011) for comparison between public and private communication in multiple audience cheap-talk models.

There is a large body of literature on information aggregation through strategic voting since the seminal contributions of Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1997, 1998), and Li, Rosen, and Suen (2001).[5] The key insight of this literature is that, despite voting simultaneously, a rational voter votes as if she knows that her vote is pivotal. In our model, under the optimal information structure, every vote for a is pivotal. Nevertheless, the sender can still increase the influence of the voters whose preferences are closer to his own.

A set of papers study the impact of voting participation cost on voting participation and outcome; See Palfrey and Rosenthal (1985), Borgers (2004), Feddersen and Sandroni (2006), Taylor and Yildirim (2010a,b), and Krishna and Morgan (2012). Unlike these models, we assume that the voting cost depends on how they vote instead of whether they vote. Downs (1957) and Fiorina (1976) are the first to study voting models where voters are motivated by non-instrumental considerations. Morgan and Várdy (2012) consider a model where a voter's action is driven by both

[5]This literature has been enriched in many dimensions: Gerardi and Yariv (2007) compare various voting rules when voters are allowed to deliberate before casting their votes. Jackson and Tan (2013) allow voters to consult experts before voting, and examine how disclosure and voting vary with different voting rules and with signal precision of the experts. Li (2001), Persico (2004), Gerardi and Yariv (2008) and Cai (2009) assume that voters (or committee members) endogenously collect their information individually.

instrumental and expressive motives. They show that even weak expressive motives may significantly affect equilibrium voting behavior and the optimal size of a voting body. In our model, voters pay a fixed cost for voting for the sender's preferred action. Levy (2007) analyzes how the reputational cost for voting for an action may depend on whether individual votes are revealed to the public.

Finally, we show that in the presence of voting cost, the sender finds it optimal to induce minimum winning coalitions to get his preferred action adopted. This result is consistent with the prediction of a large number of papers on coalition formation and vote buying. Examples include Baron and Ferejohn (1989), Koehler (1975) and Shepsle (1974).

The rest of the paper is organized as follows; we present the model in section 2; section 3 formalizes the sender's relevant information-design problem and provides some preliminary results; section 4 characterizes an optimal information structure; section 5 concludes; and omitted proofs are in the appendix.

2 Model

Voting Environment. A group of N voters needs to decide between two alternatives (or actions), denoted by $x \in \{a, b\}$. The collective decision is made according to a non-unanimity K-majority rule whereby alternative a is chosen if and only if it receives K or more votes, and $K < N$. Voters are labeled as $i = 1, ..., N$. Denote $\mathcal{N} \equiv \{1, ..., N\}$ as the set of all voters.

Payoffs. A voter's payoff depends on (i) the collective decision x, (ii) her vote $y \in \{a, b\}$, and (iii) the state of the world $\omega \in \{A, B\}$. Specifically, if b is chosen by the group, each voter i receives 0 in either state A or B. If the collective decision is a, she receives 1 when the state is A and 0 when the state is B. In addition, she pays a cost of $c_i > 0$ for voting for a, regardless of the state or the voting outcome.[6] Her payoff function is

$$u_i(x, y, \omega) = \mathbb{1}_{\{x=a,\omega=A\}} - c_i \mathbb{1}_{\{y=a\}} \tag{1}$$

where $\mathbb{1}_{\{z\}}$ is an indicator function which equals 1 if condition $\{z\}$ is true and 0 otherwise. We assume that $c_i < 1$, so that each voter i is willing to vote for a in state A if she is a dictator. The cost

[6] Here, we normalize the (state-and-outcome-independent) cost of voting for b to be zero. Our results remain as long as the b-voting cost is lower than the cost of voting for a for each voter. We assume that the voting cost is state-independent. In section 3.3, we discuss how our results will change when the voting cost may depend on the state.

can be interpreted as a reputation loss for supporting an unpopular alternative, or, when the voters are legislators, the political cost for voting against the interests of the constituents. We assume that

$$0 < c_1 < ... < c_N < 1.$$

The cost reflects a voter's "threshold of doubt" for the non-default action. Voter 1 is therefore the easiest to persuade to vote for a, while voter N is the hardest.

Voters are uncertain about the state. We assume that they share a common prior belief

$$\mu_0 = \Pr(\omega = A) = 0.5.$$

Each voter seeks to maximize her expected payoff. We assume that $c_K > 0.5$, so that fewer than K voters are willing to vote for a as a dictator at the prior belief.[7]

The aforementioned voting environment and voters' payoffs together define a *base voting game*. Specifically, the base voting game consists of (i) a K-majority rule, (ii) a state space $\{A, B\}$, (iii) the set of voters $\mathcal{N}$, (iv) for each voter $i \in \mathcal{N}$, an action space $\{a, b\}$ and a payoff function $u_i(\cdot)$ specified in expression (1), and (v) the common prior μ_0.

Information Structure. A sender, who prefers a regardless of the state, seeks to influence the voting outcome by controlling the information of the voters. An *information structure* consists of a set of finite spaces $\{S_i\}_{i\in\mathcal{N}}$ and a pair of probability distributions $\{\pi(\cdot|\omega)\}_{\omega=A,B} \subseteq \Delta(S)$, where S_i denotes the set of possible disclosures to voter i and $S = \times_{i\in\mathcal{N}} S_i$. We write π for $\{\pi(\cdot|\omega)\}_{\omega=A,B}$. An information structure (S, π) specifies a state-contingent distribution over *signal profiles* $s = (s_1, s_2, ..., s_N)$, where s_i denotes the signal observed by voter i and $\pi(s|\omega)$ denotes the probability that signal profile s is realized in state ω.

A base voting game and an information structure (S, π) jointly induce a *Bayesian voting game*, which proceeds as follows. First, voters observe the information structure (S, π). Then, state ω and signal profile s are realized. Finally, the voters simultaneously vote after observing their private signals. A mixed strategy of voter i is a function $\sigma_i : S_i \to [0, 1]$ that maps each of her private signal s_i to the probability of voting for a. A strategy profile is a Bayesian Nash equilibrium if each σ_i maximizes the expected payoff of voter i given σ_{-i}.

[7] The assumption of the prior belief being 0.5 is without loss of generality. Similarly, one can extend our analysis to the settings where voters have heterogenous prior beliefs. See Alonso and Câmara (2014) and Shimoji (2016) for persuasion models with such heterogeneity.

3 The information-Design Problem

The sender's objective is to design an information structure to maximize the probability of a being selected in the equilibrium. In general, the voting game may contain multiple Bayesian Nash equilibria. Throughout the paper, we assume that the sender can select his favorite equilibrium.[8] Since both the sender and voters prefer a in state A, the selection assumption amounts to letting the sender choose the equilibrium with the highest selection probability of a in state B, which is also the worst equilibrium to the voters among all equilibria in which a is chosen in state A.

3.1 Public Persuasion

To establish a benchmark, we first consider the case where the sender is restricted to using public signals. Formally, an information structure (S, π) is *public* if (i) $S_i = S_j$ for any $i, j \in \mathcal{N}$ and (ii) $\pi(s|\omega) = 0$ for $\omega = A, B$ and for any s such that $s_i \neq s_j$ for some $i, j \in \mathcal{N}$. For convenience, we drop the subscript i and use S to denote the set of public signals and s to denote a typical element of S. Write $\pi(s|\omega)$ for the probability that s is realized in state ω. Fix a public information structure (S, π). Let $\widetilde{S} \subseteq S$ be the set of signals upon observing which a is chosen in the sender's favorite equilibrium. For each s, the sender can have K or more votes for a only if:

$$\pi(s|A) \geq c_K (\pi(s|A) + \pi(s|B)).$$

In this case, voters' public posterior belief about the state being A is at least c_K, and voters $1, ..., K$ find it optimal to vote for a. Summing over $s \in \widetilde{S}$, we must have:

$$\sum_{s \in \widetilde{S}} \pi(s|A) \geq c_K \sum_{s \in \widetilde{S}} (\pi(s|A) + \pi(s|B)).$$

Since $\sum_{s \in \widetilde{S}} \pi(s|A) \leq 1$, $\sum_{s \in \widetilde{S}} (\pi(s|A) + \pi(s|B)) \leq 1/c_K$. It is easy to see that the bound can be achieved by a binary public signal $S = \{a, b\}$ with $\pi(a|A) = 1$ and $\pi(a|B) = (1 - c_K)/c_K$, and an equilibrium in which voters 1 to K vote for a if and only if they observe signal a while voters $K+1$ to N always vote for b. Intuitively, the sender's optimal strategy is to target voters 1 to K, who are the easiest to persuade. Since voter K is the hardest to persuade among this group, a would be chosen if and only if voter K is willing to vote for a as a dictator.

[8] The equilibrium selection assumption in our setting, requiring coordination among multiple voters, is stronger than the one in Kamenica and Gentzkow (2011), where only one receiver is involved. See Bergemann and Morris (2017) and Mathevet, Perego, and Taneva (2017) for discussion on equilibrium selection in information-design problems.

3.2 Simplifying the Sender's Problem

We now return to the general case. We first show that, by the revelation principle, we can focus on *obedience* strategies and binary information structures. An information structure (S, π) is *binary* if $S_i = \{a, b\}$ for each $i \in \mathcal{N}$. We can view $s_i \in \{a, b\}$ as a recommendation to vote for s_i. A strategy σ_i is *obedient* if it follows the recommendation (i.e., $\sigma_i(a) = 1 - \sigma_i(b) = 1$). We say that a voter votes obediently if she follows an obedience strategy, and an equilibrium is obedient if every voter's equilibrium strategy is obedient.

Lemma 1. *For any information structure (S, π), if σ is a Bayesian Nash equilibrium in the voting game induced by (S, π) and $\tau(\sigma)$ is the state-dependent distribution over vote profiles induced by σ, then every voter voting obediently is a Bayesian Nash equilibrium in the voting game induced by the binary information structure $\left(\{a, b\}^N, \tau(\sigma)\right)$.*

Lemma 1 is a consequence of the revelation principle (Myerson 1986).[9] It says that if a distribution of vote profiles is an equilibrium outcome under some information structure, then it must also be the outcome of an *obedience* equilibrium under a binary information structure where each vote profile is recommended with the profile's equilibrium realization probability in the original equilibrium. Thus, we can characterize the voting behavior that yields the highest selection probability of a under any information structure by considering only obedience equilibria. Henceforth, unless otherwise stated, all information structures are binary and, when it does not cause confusion, we drop the signal space and denote a binary information structure simply by π.

A signal profile is called *a-winning* if K or more voters are recommended to vote for a. It is an *minimum a-winning coalition* if exactly K voters are recommended to vote for a. We denote

$$S_a \equiv \{s \in \{a, b\}^N | |\{i \in \mathcal{N} : s_i = a\}| \geq K\}$$

as the set of *a-winning signal profiles*,[10] and

$$S_a^* \equiv \{s \in S_a | |\{i \in \mathcal{N} : s_i = a\}| = K\}$$

as the set of *minimum a-winning signal profiles*. Similarly, the sets of *b-winning* and *minimum b-winning signal profiles*, are denoted by S_b and S_b^*, respectively. For $x \in \{a, b\}$ and $i \in \mathcal{N}$, the sets

$$S_{i,x} \equiv \{s \in S_x | s_i = x\} \text{ and } S_{i,x}^* \equiv \left\{s \in S_x^* | s_i = x\right\}$$

[9] Also see Taneva (2014) and Bergemann and Morris (2016a, 2017) for recent application of this idea to persuasion problem.

[10] We use $|X|$ to denote the total number of elements in a finite set X.

denote the set of x-winning signal profiles and the set of minimum x-winning signal profiles where voter i is recommended to vote for x. We say that voter i is *pivotal* in signal profile s if $K-1$ voters other than voter i are recommended to vote for a in s. For each (minimum) x-winning signal profile, we call the set of voters who are recommended to vote for x as the corresponding *(minimum) x-winning coalition*.

Fix an information structure π. Consider a voter i who is recommended to vote for a with a strictly positive probability. Conditional on $s_i = a$ and assuming all voters other than i vote obediently, voter i's expected payoff for voting for a is

$$-c_i + \frac{\sum_{s\in S_{i,a}} \pi(s|A)}{\sum_{s\in S:s_i=a} (\pi(s|A)+\pi(s|B))} \tag{2}$$

where the first term is the cost of voting for a, and the second term is, conditional on i receiving recommendation a, the probability of the state being A and the collective decision being a. If she votes for b, her payoff is

$$\frac{\sum_{s\in S_{i,a}\setminus S^*_{i,a}} \pi(s|A)}{\sum_{s\in S:s_i=a} (\pi(s|A)+\pi(s|B))}. \tag{3}$$

It is therefore optimal for voter i to vote obediently for a if

$$-c_i \sum_{\omega=A,B} \sum_{s\in S:s_i=a} \pi(s|\omega) + \sum_{s\in S^*_{i,a}} \pi(s|A) \geq 0. \tag{4}$$

Similarly, it is optimal for voter i to vote obediently for b if

$$-c_i \sum_{\omega=A,B} \sum_{s\in S:s_i=b} \pi(s|\omega) + \sum_{s\in S^*_{i,b}} \pi(s|A) \leq 0. \tag{5}$$

Voting obediently is, therefore, a Bayesian Nash equilibrium if π satisfies constraints (4) and (5) for every $i \in \mathcal{N}$.

By Lemma 1, the sender's problem is to choose a probability distribution over signal profiles to maximize the selection probability of a subject to the voters' obedience constraints. For any π, let

$$Q(\pi) \equiv 0.5 \sum_{s\in S_a} (\pi(s|A)+\pi(s|B))$$

denote the total probability of implementing a when voters vote obediently. Let Π denote the set of binary information structures. The sender's information-design problem (Problem P-0) is:

$$\max_{\pi\in\Pi} Q(\pi) \tag{P-0}$$

subject to constraints (4) and (5) for every $i \in \mathcal{N}$. The following propositions establish some basic properties of the optimal information structure.

Proposition 1. *If π is a solution to Problem (P-0), then $\sum_{s \in S_a} \pi(s|A) = 1$.*

Proposition 1, which says that a is always selected in state A under any optimal solution π, results from the fact that both the sender and voters prefer a in state A.

Proposition 2. *There exists a solution π to Problem (P-0) with $\pi(s|A) = \pi(s|B) = 0$ for any $s \notin S_a^* \cup \{(b, ..., b)\}$. Furthermore, if π is a solution to Problem (P-0) and $Q(\pi) < 1$, then $\pi(s|A) = \pi(s|B) = 0$ for any $s \in S_a \setminus S_a^*$.*

Proposition 2 says that there is an optimal solution to the sender's problem where a is chosen always by exactly K votes and b is chosen always unanimously. Under this optimal information structure, when a voter is recommended to vote for a, she knows that exactly $K-1$ other voters are also recommended to vote for a; in other words, she knows that her vote is pivotal. Furthermore, when a is not always selected ($Q(\pi) < 1$), a will never receive more than K votes under any optimal information structure. Intuitively, because a voter has to pay a cost for voting for a regardless of the collective decision, she is convinced only if she is pivotal with a sufficiently high probability. It is therefore optimal for the sender to select a with a minimum number of votes. Note that when an optimal solution satisfies the restriction in Proposition 2, the obedience constraint to vote for b, condition (5) is not binding for every i, since a voter who is recommended to vote for b is never pivotal.

3.3 On Voters' Preferences

We assume that a voter's decision is driven both by the instrumental motive, which depends on the state and the voting outcome, and by the voting cost, which depends on her own vote. We make a number of assumptions to simplify the analysis. In this subsection, we discuss what happens if some of these assumptions are relaxed.

Instrumental Preference. We assume that alternatives a and b yield the same payoff in state B for all voters. Proposition 2 will continue to hold if b yields a slightly higher payoff than a in state B. However, if b yields a significantly higher payoff in state B, then a voter may be reluctant to vote for a if she believes that she may be pivotal in state B. In that case, in state A the sender will continue to induce exactly K voters to vote for a. This is because the only way for sender to motivate a voter to vote for a is to convince her that her vote is likely to be pivotal in state A. But, in state B, the sender may try to induce $K+1$ voters to vote for a to avoid making any voter pivotal. The qualitative property of the optimal information structure, however, does not change.

Voting Cost. We assume that a voter must pay a cost for voting for a regardless of the state. But in some cases, the voting cost can be state dependent. For example, a director voting for a controversial project may suffer a bigger reputation loss if the project fails. If voter i pays c_i for voting for a only when the state is B, then voter i's obedience constraint to vote for a becomes

$$-c_i \sum_{s \in S: s_i = a} \pi(s|B) + \sum_{s \in S^*_{i,a}} \pi(s|A) \geq 0. \tag{6}$$

Both Propositions 1 and 2 continue to hold, since it remains in the sender's interest to select a with the minimum number of votes in state B and to convince voters that their votes are pivotal in state A.

Additionally, we want to emphasize that Proposition 2 depends crucially on the assumption that there is no state-contingent cost for voting for b when the state is A. Suppose voters care only about voting "correctly" and not at all about the outcome. Specifically, imagine that each voter i pays $c_i < 1$ when she votes for a in state B and $d_i > c_i$ when she votes for b in state A. In this case, it is optimal for the sender to recommend each voter i to vote for a with probability one in state A and with probability c_i/d_i in state B. The sender's task is to coordinate the votes in state B to maximize the selection probability of a.

4 The Optimal Information Structure

Let

$$\Pi^{piv} \equiv \left\{\pi \in \Pi | \pi(s|A) = \pi(s|B) = 0, \forall s \notin S^*_a \cup \{(b, ..., b)\}\right\}$$

denote the set of information structures that assign zero probability to non-minimum a-winning signal profiles. By Proposition 2, we can, without loss of generality, restrict the sender's choice set to Π^{piv}. Assuming that voter i is recommended to vote for a with strictly positive probability, her obedience constraint for a, condition (4) becomes

$$\frac{\sum_{s \in S^*_{i,a}} \pi(s|A)}{\sum_{s \in S^*_{i,a}} \pi(s|A) + \sum_{s \in S^*_{i,a}} \pi(s|B)} \geq c_i \tag{7}$$

where the left-hand side is voter i's *pivotal belief*, i.e., the posterior belief conditional on being pivotal. Thus, voter i is willing to vote for a only if her pivotal belief exceeds c_i.

For the analysis that follows, it is convenient to write $l_i \equiv c_i/(1-c_i)$ as the likelihood ratio of the cutoff pivotal belief. Since l_i is strictly increasing in c_i, our assumption that c_i increases in i with

$c_K > 0.5$ means that

$$0 < l_1 < ... < l_N,$$

and $l_K > 1$. Since the obedience constraint for voting for b is not binding, we can write the sender's problem as Problem (P-1)

$$\max_{\pi \in \Pi^{piv}} \sum_{s \in S_a^*} \pi(s|B) \tag{P-1}$$

subject to

$$\sum_{s \in S_a^*} \pi(s|A) = 1; \tag{8}$$

$$\sum_{s \in S_{i,a}^*} \pi(s|A) \geq l_i \sum_{s \in S_{i,a}^*} \pi(s|B) \ \forall i \in \mathcal{N} \tag{9}$$

where the "budget" constraint (8) is a consequence of Propositions 1 and 2, and the obedience constraint (9) is immediate given condition (7). Hereafter, we let Q_B denote the value of Problem (P-1).

4.1 A Three-Voter Case

We illustrate the main features of an optimal information structure with a simple example where $N = 3$ and $K = 2$. In this case, there are three minimum a-winning signal profiles; namely, aab, aba, and baa.[11] The sender's Problem (P-1) is

$$\max_{\pi \in \Pi^{piv}} \{\pi(aab|B) + \pi(aba|B) + \pi(baa|B)\} \tag{10}$$

subject to

$$l_1[\pi(aab|B) + \pi(aba|B)] \leq \pi(aab|A) + \pi(aba|A), \tag{11}$$

$$l_2[\pi(aab|B) + \pi(baa|B)] \leq \pi(aab|A) + \pi(baa|A), \tag{12}$$

$$l_3[\pi(aba|B) + \pi(baa|B)] \leq \pi(aba|A) + \pi(baa|A), \tag{13}$$

and

$$\pi(aab|A) + \pi(aba|A) + \pi(baa|A) = 1. \tag{14}$$

Recall that when the sender is restricted to using public signals, the best he can do is to design a signal that induces voters 1 and 2 to vote for a with probability one in state A and with probability

[11] We use aab to denote the signal profile that recommends voters 1 and 2 to vote for a and voter 3 to voter for b.

$1/l_2$ in state B. In our current framework, this amounts to the information structure:

$$\begin{aligned}
\pi(baa|A) = 0 \quad &; \quad \pi(baa|B) = 0; \\
\pi(aba|A) = 0 \quad &; \quad \pi(aba|B) = 0; \\
\pi(aab|A) = 1 \quad &; \quad \pi(aab|B) = \frac{1}{l_2}.
\end{aligned}$$

Note that the obedience constraint of voter 1 is slack as $l_1 < l_2$.

The sender can increase the probability of a being selected in state B by adjusting the information structure in the following way:

1. Decrease $\pi(aab|A)$ by ϵ and increase $\pi(baa|A)$ by ϵ. Constraint (12) is still binding and constraints (11) and (13) are slack if $\epsilon > 0$ is sufficiently small.

2. Increase $\pi(aba|B)$ by $\epsilon/l_3 > 0$. Now constraints (12) and (13) are both binding. As long as ϵ is sufficiently small, constraint (11) will still be slack.

The information structure now becomes

$$\begin{aligned}
\pi(baa|A) = \epsilon \quad &; \quad \pi(baa|B) = 0; \\
\pi(aba|A) = 0 \quad &; \quad \pi(aba|B) = \frac{\epsilon}{l_3}; \\
\pi(aab|A) = 1-\epsilon \quad &; \quad \pi(aab|B) = \frac{1}{l_2}.
\end{aligned}$$

The selection probability of a in state B increases by ϵ/l_3. The shift allows the sender to take advantage of the slackness of voter 1's obedience constraint. Now, in state B voter 1 belongs to two a-winning coalitions that occur with a strictly positive probability. Note that it is crucial that voter 3 observes only the recommendation for her but not those for the other voters. If voter 3 knew that the recommendations are aba (and not baa), she would not vote for a.[12]

The following proposition describes the optimal information structure in the three-voter case.

Proposition 3. *Under the optimal information structure, the probability that a is selected in state B is*

$$Q_B = \min\left(\frac{1}{l_2}\frac{l_2+l_3}{l_1+l_3}, 1\right).$$

[12] In fact, aba is a perfect signal of state B in the sense that $\pi(aba|A) = 0$ and $\pi(aba|B) > 0$.

When $Q_B < 1$, any optimal information structure must satisfy

$$\pi(baa|A) = \frac{l_3}{l_2}\frac{l_2 - l_1}{l_1 + l_3} \quad ; \quad \pi(baa|B) = 0;$$
$$\pi(aba|A) = 0 \quad ; \quad \pi(aba|B) = \frac{1}{l_2}\frac{l_2 - l_1}{l_1 + l_3};$$
$$\pi(aab|A) = \frac{l_1}{l_2}\frac{l_3 + l_2}{l_1 + l_3} \quad ; \quad \pi(aab|B) = \frac{1}{l_2}.$$

The information structure in Proposition 3 can be obtained by choosing ϵ in the aforementioned procedure such that constraint (11) binds. It is worth to mention that voter 3, who is the hardest to persuade, votes for a with positive probability. Furthermore, it is straightforward to check that

$$Q_B = \pi(aab|B) + \pi(aba|B) > \pi(aab|B) + \pi(baa|B) > \pi(baa|B) + \pi(aba|B).$$

That is, an easier-to-persuade voter is more likely to vote for a in state B, and voter 1 is pivotal whenever a is selected in state B.

As is common in the literature, the sender finds it optimal to obfuscate the states. A more remarkable feature of the optimal information structure is that the sender also *obfuscates the winning coalitions*. By using private signals, the sender can conceal from a voter who else is being recommended to vote for a.[13] The obfuscation allows the sender to fully take advantage of the voters' heterogenous preferences—all three obedience constraints for voting for a are binding under the optimal information structure.

The gain from public to private persuasion is

$$\pi(aba|B) = \frac{l_2 - l_1}{l_1 l_2 + l_2 l_3}, \tag{15}$$

which decreases in l_3 and l_1 and vanishes as either $l_1 \to l_2$ or $l_3 \to \infty$. In the case of public persuasion, the slackness in voter 1's obedience constraint disappears at the first limit; voter 3 becomes almost impossible to persuade at the second limit, and the slackness in voter 1's obedience constraint has a very small effect.

When it is less than 1, the probability that a is adopted in state B is

$$Q_B = \pi(aba|B) + \pi(aab|B) = \frac{l_2 + l_3}{l_2\,(l_1 + l_3)},$$

[13] The randomization over multiple winning coalitions also implies that some winning coalitions contain voters who are not among the easiest to persuade, which is also present in an example in Alonso and Câmara (2016). Despite the similarity of the results, the mechanisms behind them are very different.

which is decreasing in l_i for every i: this is intuitive. Increasing l_i is equivalent to increasing a voter's threshold of doubt, c_i. To make the recommendation convincing, the signal must be more accurate about the realized state, leaving the sender less room to obfuscate states for voter i. Since all voters are involved in the optimal information structure, Q_B is decreasing in every l_i. Since $l_3 > l_2 > 1$, $Q_B < 1$ when $l_1 > 1$ or when l_3 goes to infinity. Thus, the selection probability of a can equal to one, only if voter 1 is willing to vote pivotally for a ex ante and voter 3 is not too difficult to persuade.

4.2 General Case

When there are N voters, there are $\binom{N}{K}$ minimum a-winning signal profiles. Instead of dealing with many minimum a-winning signal profiles, it is more convenient to reformulate the sender's problem in terms of the probabilities of each voter receiving a recommendation to vote for a.

Let $\alpha = (\alpha_1, ..., \alpha_N)$ and $\beta = (\beta_1, ..., \beta_N)$ be N-vectors. Suppose that there is a feasible solution π to Problem (P-1) such that, for all $i \in \mathcal{N}$,

$$\alpha_i = \sum_{s \in S^*_{i,a}} \pi(s|A); \tag{16}$$

$$\beta_i = \sum_{s \in S^*_{i,a}} \pi(s|B). \tag{17}$$

Substituting conditions (16) and (17) into voter i's obedience constraint (9) yields

$$l_i \beta_i \leq \alpha_i. \tag{18}$$

Since each minimum a-winning coalition includes K voters voting for a, the probabilities that a is chosen in states A and B are, respectively,

$$\frac{1}{K} \sum_{i=1}^{N} \alpha_i = 1, \tag{19}$$

$$\frac{1}{K} \sum_{i=1}^{N} \beta_i \leq 1. \tag{20}$$

The equality in condition (19) follows conditions (8) and (16). Since the probability that voter i belongs to a minimum a-winning coalition must be lower than the probability that a is selected, we have

$$\alpha_i \in [0,1] \ \forall i; \tag{21}$$

$$\beta_i \in \left[0, \frac{1}{K} \sum_{j=1}^{N} \beta_j\right] \ \forall i. \tag{22}$$

Since, under an optimal information structure, every vote for a is pivotal, we shall refer to a voter's probability of being recommended to vote for a as her *pivotal probability*. The discussion above means that if (α, β) are the voters' pivotal probabilities in state A and B, respectively, of a feasible solution to the sender's problem (P-1), then (α, β) must satisfy condition (18) for all $i \in \mathcal{N}$, as well as conditions (19), (20), (21), and (22). The following lemma establishes the converse.

Lemma 2. *For any (α, β) that satisfy conditions (18), (19), (20), (21), and (22), there exists a feasible solution π to Problem (P-1) such that conditions (16) and (17) hold for all $i \in \mathcal{N}$.*

Lemma 2 means that we can restate Problem (P-1) directly in terms of pivotal probabilities. Formally, consider Problem (P-2):

$$Q_B = \max_{\alpha,\beta} \frac{1}{K} \sum_{i=1}^{N} \beta_i \tag{P-2}$$

subject to conditions (18), (19), (20), (21), and (22).

The following lemma is immediate.

Lemma 3. *If π is a solution to Problem (P-1), then (α, β) defined by conditions (16) and (17) is a solution to Problem (P-2). Conversely, if (α, β) is a solution to Problem (P-2), then there exists a solution π to Problem (P-1) such that conditions (16) and (17) hold.*

Intuitively, voters use the pivotal probabilities in state A as "input" to "produce" pivotal probabilities in state B. For each unit of α_i, voter i can produce $1/l_i$ unit of β_i up to an "output ceiling" of $\sum_{j=1}^{N} \beta_j / K$. The sender's task is to divide K units of pivotal probabilities in state A among voters to maximize total production of pivotal probabilities in state B.

It is inefficient to allocate all K units of input to voters 1 to K (as would be the case if the sender uses a public signal that is designed to persuade voter K). In this case the maximum output consistent with the output ceiling is to have each voter 1 to K produce $1/l_K$ unit of β_i, meaning that some of the input allocated to voters 1 to $K-1$ are wasted.

To characterize the solution to Problem (P-2) , we first solve a relaxed problem and then relate the solution to Problem (P-2) to the solution of this relaxed problem. Consider Problem (P-3):

$$U(Y) = \frac{1}{K} \max_{\alpha,\beta} \sum_{i=1}^{N} \beta_i \tag{P-3}$$

$$\text{s.t. } l_i \beta_i \leq \alpha_i, \forall i \tag{23}$$

$$\sum_i \alpha_i = K, \alpha_i \in [0,1]; \beta_i \in [0, Y], \forall i. \tag{24}$$

Problem (P-3) discards constraint (20) in Problem (P-2), and replaces the endogenous ceiling $\sum_i \beta_i / K$ in (22) by an exogenous parameter Y in (24). Notice that in an optimal solution to Problem (P-2) , Q_B appears in constraint (22) which makes Q_B a fixed point of $U(\cdot)$.

Since l_i increases in i, the optimal solution to Problem (P-3) is straightforward. First, allocate as many units of input as possible to voter 1, who is the most efficient in producing β_1, until $\alpha_1 = \min(1, l_1 Y)$.[14] Then, allocate as many units of input as possible to voter 2 until $\alpha_2 = \min(1, l_2 Y)$, and so on. Continue the process until either all K units of input are allocated, or each voter i has received either one or $l_i Y$ units of input. Let

$$i^*(Y) \equiv \max\left\{ i \in \mathcal{N} \,\middle|\, \sum_{j=1}^{i} \min\left(1, l_j Y\right) < K \right\}$$

denote the last voter i with $\alpha_i = \min(1, l_i Y)$. The value of Problem (P-3) is

$$U(Y) = \frac{1}{K}\left\{ \sum_{j=1}^{i^*(Y)} \min\left(Y, \frac{1}{l_j}\right) + \mathbb{1}_{\{i^*(Y)<N\}} \left(K - \sum_{j=1}^{i^*(Y)} \min\left(1, l_j Y\right) \right) \frac{1}{l_{i^*(Y)+1}} \right\}, \tag{25}$$

where $\mathbb{1}_{\{i^*(Y)<N\}}$ equals one if $i^*(Y) < N$ and zero otherwise.

When $i^*(Y) < N$, all input are exhausted at the end of the allocation process. In this case, following the discussion above, the unique optimal solution to Problem (P-3) is to set, for each voter i,

$$\widehat{\alpha}_i(Y) = \begin{cases} \min(1, l_i Y) & if \quad i \le i^*(Y), \\ K - \sum_{j=1}^{i^*(Y)} \min\left(1, l_j Y\right) & if \quad i = i^*(Y)+1, \\ 0 & if \quad i > i^*(Y)+1, \end{cases}$$

and

$$\widehat{\beta}_i(Y) = \begin{cases} \min\left(Y, \frac{1}{l_i}\right) & if \quad i \le i^*(Y), \\ \frac{K - \sum_{j=1}^{i^*(Y)} \min(1, l_j Y)}{l_i} & if \quad i = i^*(Y)+1, \\ 0 & if \quad i > i^*(Y)+1. \end{cases}$$

Proposition 4. *In problem (P-2), $Q_B < 1$ if and only if $U(1) < 1$. Furthermore, $Q_B < 1$ if either $l_1 > 1$ or l_{K+1} is sufficiently large.*

The first part of Proposition 4 provides a necessary and sufficient condition for the value of Problem (P-2) to be strictly less than one. To establish the sufficiency, notice that any feasible solution to Problem (P-2) is also feasible in Problem (P-3) when $Y = 1$. Hence, if $U(1) < 1$, the value

[14] Since β_i must not exceed Y, raising α_i beyond $l_i Y$ can not further increase β_i.

of Problem (P-2) must also be strictly less than 1. We prove the necessity in the Appendix. In the three-voter example, $Q_B < 1$ when $l_1 > 1$ or l_3 is sufficiently large. The second part of Proposition 4 shows that the same is true in general. Intuitively, the sender is trying to choose an information structure that maximizes the impact of voter 1, who is the easiest to persuade in the group. If voter 1 is unwilling to vote for a as a dictator with probability one, then there is no way the sender can make the group select a with probability one. Furthermore, as we shall show in Proposition 5, under the optimal information structure at least one voter who is harder to persuade than voter K must also be persuaded to vote for a with positive probability in state B. Even when l_1 is very small, Q_B will still be less than one when this voter is very hard to persuade.

The following proposition describes some key properties of the optimal solution to Problem (P-2).

Proposition 5. *When $Q_B < 1$ and $i^*(Q_B) < N$, then $\left(\alpha^*, \beta^*\right) \equiv \left(\widehat{\alpha}(Q_B), \widehat{\beta}(Q_B)\right)$ is the unique solution to (P-2) and has the following properties:*

1. $l_i \beta_i^* = \alpha_i^* \ \forall i$,
2. $\alpha_i^* = \min(1, l_i Q_B)$ *if* $\alpha_j^* > 0$ *for some* $j > i$,
3. β_i^* *decreases in* i, $\beta_1^* = Q_B$, *and* $\beta_{K+1}^* > 0$

When $Q_B < 1$ and $i^*(Q_B) < N$, the unique solution to Problem (P-3) with $Y = Q_B$ is also a unique solution to Problem (P-2). The sender persuades voters $i = 1, ..., i^*(Q_B) + 1$. Three properties of the optimal solution follows the definition of $\left(\widehat{\alpha}(Q_B), \widehat{\beta}(Q_B)\right)$. First, each voter's obedience constraint for voting for a is binding. If the constraint for some voter i were not binding, the sender could either increase β_i or reduce α_i (and increase some other α_j). Second, the sender should allocate as many units of input as possible to voters with the lowest l_i's. If some input is allocated to voter j (i.e., $\alpha_j^* > 0$), then either the input limit (21) or output ceiling (22) must be reached for any voter $i < j$. Third, voters who are easier to persuade have higher pivotal probabilities in state B than voters who are harder to persuade. In particular, the pivotal probability of voter 1 in state B is equal to Q_B. Hence, as in the three-voter example, voter 1 must belong to every minimum a-winning coalition in state B that is assigned with positive probability. Lastly, voter $K+1$ has a strictly positive pivotal probability in state B. Since each minimum a-winning profile has only K votes for a, an optimal information structure must assign strictly positive probabilities to multiple

minimum a-winning profiles.[15]

The following proposition describes how Q_B is affected by the preferences of voters and the majority requirement.

Proposition 6. *In Problem (P-2), when $Q_B < 1$ and $i^*(Q_B) < N$,*

1. *Q_B is strictly decreasing in l_i if $\alpha_i^* > 0$,*
2. *$Q_B \to 1/l_K$ as either $l_1 \to l_K$ or $l_{K+1} \to \infty$, and*
3. *Q_B decreases in K.*

Part 1 of Proposition 6 says that Q_B increases when any voter who votes for a with a strictly positive probability becomes easier to persuade. Recall that the sender can achieve $Q_B = 1/l_K$ with public signals. Part 2 says that the gain from persuading with private signals disappears either if voters 1 to K have the same preferences, or if voters $K+1$ to N are very hard to persuade. In the first case, as in the three-voter example, when $l_1 \to l_K$, the obedience constraint (18) becomes binding for every voter i under the optimal public information structure. Thus, there is no room for manipulation. In the second case, recall that to raise Q_B above $1/l_K$, at least one voter $i > K$ must be persuaded to vote for a. When l_{K+1} goes to infinity, voters $K+1, \ldots, N$ become impossible to persuade. Lastly, part 3 says that Q_B decreases in the the number of votes needed to select a. It is well known that a group may choose a large majority requirement to encourage information acquisition when information is costly (see Li, 2001). Our results suggest that doing so may also have the benefit of limiting the influence of interested third parties. The result holds even when $l_{K+1} = l_K$. Intuitively, a larger K, in addition to forcing the sender to persuade more voters, also makes it harder to manipulate the collective decision.

Propositions 5 and 6 reveal the extent to which majority voting is susceptible to manipulation when voters have voting costs. The result contrasts sharply with the case where voters' payoffs depend solely on the outcome and not on their votes. Consider a canonical strategic voting model with purely instrumental motives as in Austen-Smith and Banks (1996). There are two states, A

[15] When $Q_B = 1$ or $i^*(Q_B) = N$, there may be multiple optimal solutions. When the voters are sufficiently easy to persuade, the sender may achieve $Q_B = 1$ without targeting the voters who are the easiest to persuade. When some voters are very easy to persuade and some very hard to persuade (e.g., some l_i close to zero and some l_i very large), $Q_B < 1$ and $i^*(Q_B) = N$. In this case, in every optimal solution $\beta_i = \min(Q_B, 1/l_i)$ for all $i \in \mathcal{N}$, but there may be multiple ways to choose α_i that are optimal.

and B, and two actions, a and b. When a is the collective choice, each voter i receives $x_i > 0$ in state A and 0 in state B. When b is the collective choice, each voter i receives 0 in state A and $y_i > 0$ in state B. It is easy to see that the following is a strictly perfect Bayesian Nash equilibrium:

When the state is B, with probability $1-\epsilon$ all voters are asked to vote for a, and with probability ϵ the sender randomly selects one of the minimum b-winning profiles with equal probability. When the state is A, *the sender randomly selects one of the minimum a-winning profiles with equal probability.*

In this equilibrium, a is chosen with probability $1-0.5\epsilon$. Intuitively, the sender can get a selected with probability arbitrarily close to one by exploiting a non-pivotal voter's indifference between voting for or against a.[16] In our model, since the voting cost must be paid regardless of the outcome of the vote, a voter is unwilling to vote for a unless she believes that there is a sufficiently high chance that her vote is pivotal and the state is A.

The comparison above suggests an advantage of making the voting process more transparent. When a voting body is subject to outside manipulation, the reputational concern of voters prevents the collective decision from being fully manipulated. In a secret ballot, it is hard to infer how each individual voter votes, which weakens the reputational concern.

5 Conclusion

This paper considers the information-design problem of a sender who tries to persuade a group of voters. We characterize the optimal information structure and explain why private persuasion strictly outperforms public persuasion.

We assume that the sender has no private information about the state before choosing the information structure. It is natural to ask what happens if the assumption is relaxed, especially in our voting context where an interested party or politician engaged in persuading has superior information about the state to voters. Alonso and Câmara (2016) use the results from Alonso and Câmara (2018) to argue that, if the sender knows the state before persuasion, then the sender's favorite equilibrium in this informed-sender game is an equilibrium in which all informed senders pool on the same signal used by an uninformed sender. Their result extends to the current paper. That is, if the information designer knows the true state before choosing the information structure, there is an equilibrium in this informed-designer game where all types of senders pool on

[16] Also see Bardhi and Guo (2018).

the same signal as the uninformed sender. It is also natural to ask what happens if voters have private information, as in Kolotilin, Mylovanov, Zapechelnyuk, and Li (2017), Kolotilin (2018) and Bergemann, Bonatti, and Smolin (2018). We leave this extension for future work.

A Appendix: Omitted Proofs

Proof of Lemma 1. The proof follows directly from the Revelation Principle as in Myerson (1986), Bergemann and Morris (2016a) and Taneva (2014) and is omitted. □

Proof of Proposition 1. Suppose that $\pi(s|A) > 0$ for some $s \in S_b$. Because $c_i \in (0,1), \forall i$, the sender can always set $\pi(s|A) = 0, \forall s \in S_b$ and increase the probabilities that $\sum_{s \in S_a} \pi(s|A)$ without violating voters' obedience constraints. □

Proof of Proposition 2. For any $s \in S_a \setminus S_a^*$, we can pick some $s' \in S_a^*$ such that for all $i \in \mathcal{N}$, $s'_i = a$ only if $s_i = a$. Hence, we can construct a function $f(s) : S_a \setminus S_a^* \to S_a^*$ such that $f_i(s)$, the i-th component of $f(s)$, is equal to a only if $s_i = a$. In words, $f(s)$ is a minimum a-winning profile where every voter who is recommended to vote for a is also recommended to vote for a in s.

Let f^{-1} denote the inverse mapping of f. Suppose π is an optimal solution to Problem (P-0). By Proposition 1, $\sum_{s \in S_a} \pi(s|A) = 1$. Define π' as follows:

$$\pi'(s|A) \equiv \begin{cases} \pi(s|A) + \sum_{s' \in f^{-1}(s)} \pi\left(s'|A\right) & if \quad s \in S_a^*; \\ 0 & if \quad s \notin S_a^*; \end{cases}$$

$$\pi'(s|B) \equiv \begin{cases} \pi(s|B) + \sum_{s' \in f^{-1}(s)} \pi\left(s'|B\right) & if \quad s \in S_a^*; \\ \sum_{s' \in S_b} \pi\left(s'|B\right) & if \quad s = (b, ..., b); \\ 0 & if \quad s \notin S_a^* \cup \{(b, ..., b)\}. \end{cases}$$

Thus, π' is formed from π by shifting, in both states A and B, the probability weight from each $s \in S_a \setminus S_a^*$ to $f(s)$ and, in state B, from each $s \in S_b$ to $(b, ..., b)$. Furthermore, it is easy to check that

$$\pi'(s|A) = 0 \ \forall s \notin S_a^* \cup \{(b, ..., b)\}. \tag{26}$$

Because under π', in state A, a is always recommended to exactly K voters, a voter who is recommended to vote for b knows that her vote is never pivotal when the state is A. This means that, for each voter i, $\sum_{s \in S_{i,b}^*} \pi(s|A) = 0$ and the obedience constraint for voting for b, condition (5), is satisfied under π'.

We now turn to the obedience constraint for voting for a, condition (4), which can be rewritten as:

$$-c_i \sum_{s\in S:s_i=a} \pi(s|B) - c_i \sum_{s\in S_{i,a}\setminus S^*_{i,a}} \pi(s|A) + (1-c_i) \sum_{s\in S^*_{i,a}} \pi(s|A) \geq 0. \tag{27}$$

By supposition, π satisfies (4) for all voters i. By construction,

$$\sum_{s\in S_{i,a}\setminus S^*_{i,a}} \pi'(s|A) \leq \sum_{s\in S_{i,a}\setminus S^*_{i,a}} \pi(s|A),$$

and

$$\sum_{s\in S^*_{i,a}} \pi'(s|A) \geq \sum_{s\in S^*_{i,a}} \pi(s|A).$$

Furthermore, since $f_i(s) = a$ only if $s_i = a$, for each voter i,

$$\sum_{s\in S:s_i=a} \pi'(s|B) \leq \sum_{s\in S:s_i=a} \pi(s|B).$$

It follows that π' also satisfies (4) for all voters i. Thus, π' is feasible for Problem (P-0). By construction, the probability that a is selected under π' is the same as under π. Hence, π' is also optimal. This proves the first part of the Proposition.

For the second part, note that if $\pi(s|A) > 0$ for some $s \in S_a \setminus S^*_a$, then the obedience constraint for voting for a for every voter i in the a-winning coalition in $f(s)$ will be slack under π'. If a is not the selection with probability one in state B under π', we can increase the probability of a being selected by slightly increasing $\pi'\left(f(s)|B\right)$ (while simultaneously reducing that of $\pi'(s|B)$) without violating the obedience constraint of any voter. This proves that neither π nor π' is optimal. □

Proof of Proposition 3. Since the purpose of the 3-voter example is to illustrate the idea of the more general N-voter model, we use the notations introduced in section 4.2 to prove this proposition. Note that in the three-voter case, $i^*(Q_B) < 3$ for any Q_B as $l_1 > 0$. Hence, when $Q_B < 1$, by Proposition 5, $\beta^*_1 = Q_B$ and $\alpha^*_2 = \min(1, l_2 Q_B) = 1$. It follows that $\pi(baa|B) = \pi(aba|A) = 0$. The solution is the only one that makes (11),(12), (13), and (14) binding. It follows that $Q_B = \min\{\pi(aba|B) + \pi(aab|B), 1\}$. □

Proof of Lemma 2. We show that for any α such that conditions (19) and (21) hold, there exists a $\pi(s|A)$ such that condition (16) holds.

Let $h = \binom{N}{K}$ be the number of minimum a-winning signal profiles. It is convenient to state the proposition in matrix form. Let $s^{[1]}, ..., s^{[h]}$ be an order of the minimum a-winning signal profiles.

Let W be a $N \times h$ matrix with

$$W_{ij} = \begin{cases} 1 & if \quad s_i^{[j]} = a, \\ 0 & if \quad s_i^{[j]} = b. \end{cases}$$

We need to show that there is a vector

$$\pi_A = \begin{pmatrix} \pi_{1A} \\ \vdots \\ \pi_{hA} \end{pmatrix},$$

such that $\sum_{i=1}^{h} \pi_{iA} = 1$ and $\pi_{iA} \in [0,1]$, and

$$W\pi_A = \alpha \equiv \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}.$$

Suppose by way of contradiction that no such π_A exists. By Farkas' lemma, there exists a vector

$$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{pmatrix},$$

such that

$$W^T\lambda \geq 0; \tag{28}$$

$$\alpha^T\lambda < 0. \tag{29}$$

Then each row of W^T corresponds to a signal profile where exactly K voters observe a, including one that begins with $K+1$ ones followed by $N-K$ zeros. Thus, condition (28) implies that

$$\sum_{i=1}^{K} \lambda_i \geq 0.$$

Since the player ordering can be arbitrarily rearranged, we can assume without loss of generality that λ_i is ascending in i. Hence

$$\begin{aligned} & \min_{x_i} \sum_{i=1}^{N} \lambda_i x_i \text{ s.t. } x_i \in [0,1]\, \forall i,\ \sum_{i=1}^{N} x_i = K\,, \\ = & \sum_{i=1}^{K} \lambda_i \geq 0. \end{aligned}$$

Since $\alpha_i \in [0,1]$ for all i, and $\sum_{i=1}^{N} \alpha_i = K$,

$$\sum_{i=1}^{N} \lambda_i \alpha_i \geq 0,$$

which contradicts condition (29).

Similarly, one can show that for any $\beta = (\beta_1, ..., \beta_n)$ subject to conditions (20) and (22), there exists $\pi(s|B)$ such that condition (17) holds. □

Proof of Proposition 4. We first show that if $Q_B < 1$ then $U(1) < 1$. To do so, suppose that $U(1) \geq 1$, and we shall show that $Q_B = 1$. There are two cases. In the case where $U(1) = 1$, the corresponding optimal solution to Problem (P-3) with $Y = 1$ is feasible to Problem (P-2) with a value $Q_B = 1$.

Consider the case where $U(1) > 1$, and denote (α^*, β^*) as an optimal solution. Note that both the objective function and constraints in Problem (P-3) is linear in (α, β) (for any Y). That is, if both (α, β) and (α', β') are feasible with values v and v', respectively, then for any $\lambda \in [0,1]$, $(\lambda\alpha + (1-\lambda)\alpha', \lambda\beta + (1-\lambda)\beta')$ is feasible with value $\lambda v + (1-\lambda) v'$. We know that (α, β) where

$$\alpha_i = \begin{cases} 1 & if \quad i \leq K \\ 0 & if \quad i > K \end{cases}, \beta_i = \begin{cases} 1/l_K & if \quad i \leq K \\ 0 & if \quad i > K, \end{cases}$$

is a feasible choice to Problem (P-3) with $Y = 1$ with value equal to $1/l_K < 1$. Therefore, there exists a $\lambda \in [0,1]$ such that $(\lambda\alpha + (1-\lambda)\alpha^*, \lambda\beta + (1-\lambda)\beta^*)$ is a feasible solution to Problem (P-3) with $Y = 1$ with value exactly equal to 1. This solution is also feasible Problem (P-2), meaning that $Q_B = 1$. □

To show that $Q_B < 1$ if $l_1 > 1$, note that

$$l_1 \sum_{i \in \mathcal{N}} \beta_i \leq \sum_{i \in \mathcal{N}} l_i \beta_i \leq \sum_{i \in \mathcal{N}} \alpha_i = K. \tag{30}$$

The second inequality in condition (30) is obtained by summing condition (18) over i. Dividing condition (30) by K and substituting Q_B for $\sum_{i \in \mathcal{N}} \beta_i / K$, we have

$$l_1 Q_B \leq 1.$$

We now show that $Q_B < 1$ if l_{K+1} is sufficiently large. Since $l_K > 1$, and $\min(1/l_i, 1) \leq 1$,

$$U(1) \leq \frac{1}{K}\left\{K - 1 + \frac{1}{l_K} + \frac{K}{l_{K+1}}\right\}.$$

Hence,

$$\begin{aligned}\lim_{l_{K+1}\to\infty} U(1) &\leq \lim_{l_{K+1}\to\infty} \frac{1}{K}\left\{K-1+\frac{1}{l_K}+\frac{K}{l_{K+1}}\right\}\\ &= \frac{1}{K}\left\{K-1+\frac{1}{l_K}\right\}\\ &< 1.\end{aligned}$$

Proof of Proposition 5. Suppose that $Q_B < 1$, $i^*(Q_B) < N$ and (α, β) is a solution to Problem (P-2). Notice that (α, β) must be feasible in Problem (P-3) when $Y = Q_B$, leading to $U(Q_B) \geq Q_B$. We want to show that $U(Q_B) = Q_B$. Suppose, by way of contradiction, that $U(Q_B) > Q_B$ in Problem (P-3). This is possible only if $U(Q_B) > 1$; otherwise, Q_B cannot be the value of Problem (P-2). Since $U(1) \geq U(Q_B)$, if $U(Q_B) > 1$, then $U(1) > 1$, which, by Proposition 4, implies that $Q_B = 1$, a contradiction. Hence, in this case an optimal solution to Problem (P-2) must also be optimal in Problem (P-3) with $Y = Q_B$. Since we have already shown in text that when $i^*(Q_B) < N$, that $\left(\widehat{\alpha}(Q_B), \widehat{\beta}(Q_B)\right)$ is the unique solution to Problem (P-3) with $Y = Q_B$, it must also be the unique solution to Problem (P-2).

The first and second properties of the solution follows immediately from the definition of $\left(\widehat{\alpha}(Q_B), \widehat{\beta}(Q_B)\right)$. The third property that β_i^* decreases in i follows from the first two properties and the fact that l_i is increasing. By the first two properties, we have

$$\beta_1^* = \frac{\alpha_1^*}{l_1} = \min\left(\frac{1}{l_1}, Q_B\right).$$

We want to show that $\beta_1^* = Q_B < 1/l_1$. Suppose not. Then $\beta_1^* = \min(\frac{1}{l_1}, Q_B) = \frac{1}{l_1} \leq Q_B$. Since l_i is increasing in i, $\left(\alpha_i^*, \beta_i^*\right)$ would be equal to $(1, 1/l_i)$ for $i \leq K$ and $(0,0)$ for $i > K$. But then

$$Q_B = \frac{1}{K}\sum_{i=1}^{K}\frac{1}{l_i} < \frac{1}{l_1},$$

contradicting the supposition that $1/l_1 \leq Q_B$. This proves that $\beta_1^* = Q_B < 1/l_1$. Finally, since $\alpha_1^* = l_1 Q_B < 1$, $\sum_{i=1}^{K}\alpha_i^* < K$. It then follows from the fact that $\sum_{i=1}^{N}\alpha_i^* = K$ that α_{K+1}^* and β_{K+1}^* are strictly positive. □

Proof of Proposition 6. **Part 1.** Suppose l_i decreases for some i where $\alpha_i^* > 0$. Then voter i's obedience constraint for voting for a becomes slack. Since by assumption $i^*(Q_B) < N$, we can reduce α_i^* by ϵ, increase $\alpha_{i^*(Q_B)+1}^*$ by ϵ and $\beta_{i^*(Q_B)+1}^*$ by $\epsilon / l_{i^*(Q_B)+1}$.

Part 2. Since $1/l_K$ can always be achieved under public persuasion, $Q_B \geq 1/l_K$. For any $Y > 1/l_K$, $\left(K - \sum_{j=1}^{K} \min\left(1, l_j Y\right)\right) < 0$ when l_1 (and therefore also $l_2, ..., l_{K-1}$) is sufficiently close to l_K. Hence,

$$\lim_{l_1 \to l_K} U(Y; l_1, ..., l_N) < \frac{1}{K} \sum_{i=1}^{K} \min\left(Y, \frac{1}{l_i}\right) < Y.$$

As we argued in Proposition 5, $U(Q_B) = Q_B$. Since for all $Y > 1/l_K$,

$$\lim_{l_1 \to l_K} U(Y; l_1, ..., l_N) < Y,$$

it follows that

$$\lim_{l_1 \to l_K} Q_B \leq 1/l_K.$$

A similar argument applies when l_{K+1} (and therefore $l_{K+2}, ..., l_N$) becomes sufficiently large, i.e.,

$$\lim_{l_{K+1} \to \infty} Q_B \leq 1/l_K,$$

as for all $Y > 1/l_K$,

$$\begin{aligned} & \lim_{l_{K+1} \to \infty} U(Y; l_1, ..., l_N), \\ \leq \quad & \frac{1}{K}\left[\sum_{i=1}^{K-1} \min\left(Y, \frac{1}{l_j}\right) + \frac{1}{l_K} + \lim_{l_1 \to l_K}\left(K - \sum_{j=1}^{K} \min\left(1, l_j\right)\right)\frac{1}{l_{K+1}}\right], \\ = \quad & \frac{1}{K}\left(\sum_{i=1}^{K-1} \min\left(Y, \frac{1}{l_j}\right) + \frac{1}{l_K}\right) < Y. \end{aligned}$$

Part 3. Suppose that $Q_B < 1$ and $i^*(Q_B) < N$. Recall that $i^*(Q_B) > K$ is the last voter who votes for a with probability one in state A. Suppose K increases by 1. Keeping Q_B constant for now. From Proposition 5, we know that we will allocate the new unit of pivotal probability to voter $i^*(Q_B) + 1$ and, probably, voter $i^*(Q_B) + 2$ (if $i^*(Q_B) + 1 < N$). This will increase the sum of all β_i^* by less than Q_B since, from Proposition 5, β_i is decreasing in i and $\beta_{i^*(Q_B)}^* = 1/l_{i^*(Q_B)} < Q_B$. This means that the new average (now with denominator $K+1$) will be strictly less than Q_B. Hence Q_B is no longer the solution. Since $U(Y)$ is monotone in Y, the new solution must be smaller. □

References

ALONSO, R., AND O. CÂMARA (2014): "Persuading skeptics and reaffirming believers," *Working Paper, LSE and USC.*

——— (2016): "Persuading voters," *American Economic Review*, 106(11), 3590–3605.

——— (2018): "On the value of persuasion by experts," *Journal of Economic Theory*, 174, 103–123.

ARIELI, I., AND Y. BABICHENKO (2016): "Private bayesian persuasion," *Working Papers, Israel Institute of Technology*.

AUSTEN-SMITH, D., AND J. S. BANKS (1996): "Information aggregation, rationality, and the Condorcet jury theorem," *American Political Science Review*, 90(01), 34–45.

BARDHI, A., AND Y. GUO (2018): "Modes of Persuasion Toward Unanimous Consent," *Theoretical Economics*, forthcoming.

BARON, D. P., AND J. A. FEREJOHN (1989): "Bargaining in legislatures," *American political science review*, 83(4), 1181–1206.

BERGEMANN, D., A. BONATTI, AND A. SMOLIN (2018): "The design and price of information," *American Economic Review*, 108(1), 1–48.

BERGEMANN, D., AND S. MORRIS (2013): "Robust predictions in games with incomplete information," *Econometrica*, 81(4), 1251–1308.

——— (2016a): "Bayes correlated equilibrium and the comparison of information structures in games," *Theoretical Economics*, 11(2), 487–522.

——— (2016b): "Information Design and Bayesian Persuasion Information Design, Bayesian Persuasion, and Bayes Correlated Equilibrium," *The American Economic Review*, 106(5), 586–591.

——— (2017): "Information design: A unified perspective," *Working Paper, Yale University*.

BORGERS, T. (2004): "Costly voting," *American Economic Review*, 94(1), 57–66.

BROCAS, I., AND J. D. CARRILLO (2007): "Influence through ignorance," *The RAND Journal of Economics*, 38(4), 931–947.

CAI, H. (2009): "Costly participation and heterogeneous preferences in informational committees," *The RAND Journal of Economics*, 40(1), 173–189.

DOWNS, A. (1957): *An Economic Theory of Democracy*. New York: Harper.

FARRELL, J., AND R. GIBBONS (1989): "Cheap talk with two audiences," *The American Economic Review*, pp. 1214–1223.

FEDDERSEN, T., AND W. PESENDORFER (1997): "Voting behavior and information aggregation in elections with private information," *Econometrica*, 65(5), 1029–1058.

——— (1998): "Convicting the innocent: The inferiority of unanimous jury verdicts under strategic voting," *American Political Science Review*, 92(01), 23–35.

FEDDERSEN, T., AND A. SANDRONI (2006): "A theory of participation in elections," *American Economic Review*, 96(4), 1271–1282.

FIORINA, M. P. (1976): "The voting decision: instrumental and expressive aspects," *The Journal of Politics*, 38(2), 390–413.

GERARDI, D., AND L. YARIV (2007): "Deliberative voting," *Journal of Economic Theory*, 134(1), 317–338.

——— (2008): "Information acquisition in committees," *Games and Economic Behavior*, 62(2), 436–459.

GOLDSTEIN, I., AND C. HUANG (2016): "Bayesian persuasion in coordination games," *The American Economic Review*, 106(5), 592–596.

GOLTSMAN, M., AND G. PAVLOV (2011): "How to talk to multiple audiences," *Games and Economic Behavior*, 72(1), 100–122.

INOSTROZA, N., AND A. PAVAN (2018): "Persuasion in Global Games with Application to Stress Testing," *Working Paper, Northwestern University*.

JACKSON, M. O., AND X. TAN (2013): "Deliberation, disclosure of information, and voting," *Journal of Economic Theory*, 148(1), 2–30.

KAMENICA, E., AND M. GENTZKOW (2011): "Bayesian persuasion," *American Economic Review*, 101(6), 2590–2615.

KOEHLER, D. H. (1975): "Legislative coalition formation: The meaning of minimal winning size with uncertain participation," *American Journal of Political Science*, pp. 27–39.

KOLOTILIN, A. (2018): "Optimal information disclosure: A linear programming approach," *Theoretical Economics*, 13(2), 607–635.

KOLOTILIN, A., T. MYLOVANOV, A. ZAPECHELNYUK, AND M. LI (2017): "Persuasion of a privately informed receiver," *Econometrica*, 85(6), 1949–1964.

KRISHNA, V., AND J. MORGAN (2012): "Voluntary voting: Costs and benefits," *Journal of Economic Theory*, 147(6), 2083–2123.

LEVY, G. (2007): "Decision making in committees: Transparency, reputation, and voting rules," *American economic review*, 97(1), 150–168.

LI, H. (2001): "A theory of conservatism," *Journal of Political Economy*, 109(3), 617–636.

LI, H., S. ROSEN, AND W. SUEN (2001): "Conflicts and common interests in committees," *The American Economic Review*, 91(5), 1478–1497.

MATHEVET, L., J. PEREGO, AND I. TANEVA (2017): "On Information Design in Games," *Working Paper, NYU*.

MORGAN, J., AND F. VÁRDY (2012): "Mixed motives and the optimal size of voting bodies," *Journal of Political Economy*, 120(5), 986–1026.

MYERSON, R. B. (1986): "Multistage games with communication," *Econometrica: Journal of the Econometric Society*, pp. 323–358.

PALFREY, T. R., AND H. ROSENTHAL (1984): "Participation and the provision of discrete public goods: a strategic analysis," *Journal of public Economics*, 24(2), 171–193.

——— (1985): "Voter participation and strategic uncertainty," *American Political Science Review*, 79(1), 62–78.

PERSICO, N. (2004): "Committee design with endogenous information," *The Review of Economic Studies*, 71(1), 165–191.

RAYO, L., AND I. SEGAL (2010): "Optimal information disclosure," *Journal of Political Economy*, 118(5), 949–987.

SCHNAKENBERG, K. E. (2015): "Expert advice to a voting body," *Journal of Economic Theory*, 160, 102–113.

SHEPSLE, K. A. (1974): "On the size of winning coalitions," *American Political Science Review*, 68(2), 505–518.

SHIMOJI, M. (2016): "Rationalizable Persuasion," *Working Paper, University of York*.

TANEVA, I. (2014): "Information design," *Working Paper, University of Edinburgh*.

TAYLOR, C. R., AND H. YILDIRIM (2010a): "Public information and electoral bias," *Games and Economic Behavior*, 68(1), 353–375.

——— (2010b): "A unified analysis of rational voting with private values and group-specific costs," *Games and Economic Behavior*, 70(2), 457–471.

WANG, Y. (2015): "Bayesian persuasion with multiple receivers," *Working Paper, Xiamen Univeristy*.