# Self-Evident Events and the Value of Linking*

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**Abstract**

We propose a theory of linking in long-term relationships based on what information becomes self-evident in equilibrium at the end of a stage game. We obtain a tight bound on the average per-period efficiency loss that must be incurred to enforce a stage-game outcome throughout a $T$-period repeated game when $T$ is large. Our results apply to all monitoring structures and strategy profiles. They encompass the inefficiency result in Abreu, Milgrom, and Pearce (1991), as well as the approximate-efficiency results in Compte (1998), Obara (2009), and Chan and Zhang (2016).

## 1 Introduction

Providing incentives is costly when actions are imperfectly monitored. As firms react more rapidly to information, the cost of imperfect monitoring may become so high that renders collusion unsustainable (Sannikov and Skrzypacz, 2007).[1] In a seminal paper, Abreu, Milgrom, and Pearce (1991) show that players can reduce the cost of imperfect

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[1]See also Sannikov and Skrzypacz (2010).

monitoring by linking incentives across periods if the release of monitoring information can be delayed. Subsequent research has shown that in repeated games of private monitoring a similar result can be obtained when players delay sharing their private signals endogenously (Compte, 1998, Obara, 2009, and Chan and Zhang, 2016). More recently, Rahman (2014) and Sugaya (2017a,b) explore how long-run efficiency can be attained through linking in games of public monitoring when players can manufacture private information through correlated strategies.[2]

These results suggest that the release and distribution of information in an organization can have a profound impact on the long-run incentives and efficiency of the organization. The goal of this paper is to provide a unified framework that makes precise this relationship. We characterize the value of linking in a $T$-period contracting game between a principal and a group of players. The principal's objective is to design a contract to enforce a particular stage-game action profile throughout the game, subject to the constraint that the total payments to the players be negative. The target action profile may be pure or correlated. In the latter case we assume the players may use a correlating device to coordinate their actions. Since Compte (1998), this problem has become a crucial building block in the theory of repeated game with private monitoring. Working with the $T$-period contracting problem allows us to focus on the mechanism of linking and abstract away from the problem of implementing transfers through continuation strategies.[3] The existing literature has focused on sufficient conditions on monitoring structures that ensure approximate efficiency. Our results, by contrast, apply to all monitoring structures. They explain why some outcomes can be implemented with little long-term efficiency loss and why some cannot. Furthermore, for outcomes that cannot be implemented efficiently, we provide a tight bound on the long-term efficiency loss.

A central concept in our analysis is the notion of self-evident event, which is introduced by Aumann (1976) to describe beliefs in incomplete-information games. An event is a set of states; it is self-evident if it is common knowledge whenever it occurs.

[2]Going beyond repeated games, linking also explains why it is easier to sustain collusion when firms compete in multiple markets (Matsushima, 2001), and why a principal may prefer an efficiency-wage contract that only punishes a worker who performs poorly in every period (Fuchs, 2007).

[3]Our results can be readily applied to repeated games with side-payments. See Section 8.

In our model, players play a stage game for $T$ periods. In each period, each player receives a recommendation from a correlating device before choosing an action privately, and observes a private signal after all players have chosen their actions.[4] A state in our model is thus a profile of recommendations and signals. But unlike Aumann (1976), the distribution of states in our model depends on the actions of the players. Our innovation is to apply the notion of self-evident events to the players' *equilibrium* beliefs. A set of profiles of recommendations and signals is self-evident in equilibrium if, whenever a profile in the set occurs, it is common knowledge that the realized profile is in the set.

We show that the value of linking is closely tied to the information that becomes self-evident in equilibrium during the course of the contract. Formally, a contract is a function that maps the recommendations of the randomization device and the private signals of the players into a vector of payments to the players. The contract is ex post inefficient if the total payment is strictly negative. For any outcome and any short-term contract that enforces the outcome, we can compute the expected efficiency loss of the contract conditional on an irreducible self-evident (assuming that the players follow the recommendations). Define the efficiency loss due to the self-evident component of a contract as the expected efficiency loss of the contract minus the minimum efficiency loss conditional on any irreducible self-evident event. We show that the long-run efficiency loss for implementing an outcome must be greater than the efficiency loss of self-evident component of any short-term contract that enforces the outcome. More importantly, we show that the bound is tight when the contract length is long under a *no-free-information* condition that requires that no player can deviate to acquire more information in the Blackwell sense without lowering his stage-game payoff. The condition is closely related to but is weaker and more intuitive than the condition of strict enforceability. Under a non-stationary contract that links incentives, how an action is rewarded depends on past outcomes. The no-free-information condition ensures that no player can deviate to take advantage of this dependence. Our results apply to all monitoring structures (both public and private) and action profiles (both pure and correlated). Note that, by definition, the efficiency loss due to the self-evident component

[4] If the target action profile is pure, the correlating device will always make the same recommendation to each player.

of a contract is zero if the set of all profiles of recommendations and signals is the only (and hence irreducible) self-evident event. It follows immediately that any outcome that satisfies the no-free-information condition and induces a distribution with no proper self-evident subset can be implemented approximately efficiently in the long run.

Our results build on the seminal work of Abreu, Milgrom, and Pearce (1991). They show that in symmetric repeated games, the value of linking is constrained by the need to provide separate incentives in continuation games that follow different public signals. For example, consider a binary public signal that is either "good" or "bad." Since, once realized and publicly revealed, it is common knowledge whether the signal is good or bad, the incentives after a good signal will not have any effect on the behavior in the continuation game after a bad signal. The concept of self-evident events generalizes the idea of a public event to games with private information. While a self-evident event is not literally public, in equilibrium it is as if there is an additional public signal that reveals which self-evident event has occurred. As in Abreu, Milgrom, and Pearce (1991), separate incentives must be provided in continuation games following different self-evident events. This need to provide separate incentives imposes the lower bound on the long-run efficiency loss. Our sufficiency result further shows that any residual efficiency loss that is not self-evident (in a sense we will make precise) can be reduced through linking in a long-term contract.

Our paper is closely related to the literature of repeated games with private monitoring and communication (Compte, 1998; Obara, 2009; Chan and Zhang, 2016; Fudenberg, Levine, and Maskin, 1994; Kandori and Matsushima, 1998). Compte (1998), Obara (2009), and Chan and Zhang (2016) apply the linking idea to obtain folk theorems.[5] Unlike the players in Abreu, Milgrom, and Pearce (1991) where signals arrive with a lag, the players in these models observe private signals at the end of every period. While the players may delay revealing their signals, each may nevertheless update his beliefs about other players' signals on the basis of his own. Compte (1998) and Obara (2009) impose restrictions on the signal structure to ensure that no player can

[5]The literature of repeated games with private monitoring and without communication also exploits the idea of linking. See, e.g., Matsushima (2004) and Fong, Gossner, Hörner, and Sannikov (2011).

learn about his own continuation payoffs.[6] Our approach, instead, relies on showing that, when $T$ is large, conditional on an irreducible self-evident event, there is always some player who does not update his belief significantly. Fong, Gossner, Hörner, and Sannikov (2011) pioneer this approach in the context of Prisoners' Dilemma. Chan and Zhang (2016) establish essentially a two-player version of our result.[7] Compte (1998), Obara (2009) and Chan and Zhang (2016) consider pure action profile and assume that the signal distribution has full support, which rules out public signals. Our framework applies to all monitoring structures (both public and private) and action profiles (both pure and correlated). It encompasses both the inefficiency result in Abreu, Milgrom, and Pearce (1991), as well as the approximate-efficiency results in Compte (1998), Obara (2009), and Chan and Zhang (2016).

Fudenberg, Levine, and Maskin (1994) and Kandori and Matsushima (1998) identify conditions that ensure that an outcome can be enforced by a contract with zero-sum transfers. Rahman and Obara (2010) show that such contract exists if and only if every deviating strategy is attributable. We extend Rahman and Obara (2010) in two ways. First, we show that an outcome can be enforced in the long-run almost efficiently if every deviating strategy is either attributable or detectable by action-signal profiles within some irreducible self-evident event. Furthermore, we characterize the long-run efficiency loss when some deviations are neither attributable nor detectable with respect to an irreducible self-evident event.

Since our results characterize the efficiency loss for any signal structure, they provide a transparent way to determine how a change in the information structure may improve efficiency. For example, our results suggest that the principal may improve the long-run efficiency of an organization by relying a noisy performance measure that reduces the number of self-evident events. Thus, the incentive structure of an organization may depend systematically different on the planning horizon of the organization. Furthermore, players may change the information structure by adopting a correlated strategy. The idea of using correlated strategies to enhance efficiency is due to Rahman

[6]Compte (1998) assumes independent signals. Obara (2009) considers correlated signals and identifies a condition on the signal distribution that ensures that no player can learn about his own transfer.

[7]Chan and Zhang (2016) consider transfer schemes in $n$-player games whereby each player's transfer is the sum of the other players' transformed signals. Hence, each player pair can be dealt with separately.

(2014). His approach requires that the outcome be *conditionally identifiable*. As an application of our results, we introduce a different approach that dispenses with this requirement. In particular, we show that for any strictly enforceable outcome, there always exists an $\varepsilon$-close correlated action profile that can be implemented approximately efficiently in the long run.

In two recent papers, Sugaya (2017a,b) derives upper and lower bounds in equilibrium payoffs in repeated games with private monitoring and correlated strategies. In particular, Sugaya (2017a) shows that approximate efficiency can be achieved when the players can observe their own payoffs. Sugaya's method always requires a correlating device, while ours requires one only when the outcome is correlated. While Sugaya (2017a,b) focuses on the equilibrium payoff set, we focus on the minimum efficiency loss associated with the enforcement of a given outcome.

The rest of the paper is organized as follows. The next section introduces the formal model. Section 3 reviews the results of Abreu, Milgrom, and Pearce (1991). Section 4 uses the notion of self-evident events to establish a lower bound on the long-run cost of imperfect monitoring. Section 5 shows that the bound is tight under the no-free-information condition. Section 6 characterizes the long-run efficiency loss in enforcing an action profile. Section 7 shows that any strictly enforceable action profile is virtually enforceable with almost no efficiency loss. Section 8 discusses how our results can be applied to repeated games. Section 9 concludes.

# 2 Model

## 2.1 Stage game

Consider a finite stage game endowed with a correlating device. Let $N = \{1, 2, \ldots, n\}$ denote a set of players, $A = A_1 \times \cdots \times A_n$ a finite set of action profiles, $\eta \in \Delta(A)$ a distribution over $A$, and $g = (g_1, \ldots, g_n) : A \to \mathbf{R}^n$ a profile of stage-game payoff functions. In each period, the correlating device draws $\widetilde{a} = (\widetilde{a}_1, \ldots, \widetilde{a}_n) \in A$ according to $\eta$ and privately recommends $\widetilde{a}_i$ to each player $i$. After learning $\widetilde{a}_i$, each player $i \in N$ privately chooses $a_i \in A_i$. Player $i$'s expected stage-game payoff is $g_i(a)$, where

$a = (a_1, \ldots, a_n)$. The players do not directly observe the stage-game payoffs. Instead, each player $i$ observes a signal $y_i$. The signal profile $y = (y_1, \ldots, y_n)$ is drawn from a finite set $Y = Y_1 \times \cdots \times Y_n$ according to a distribution $p(\cdot|a) \in \Delta(Y)$.

Since the only function of the correlating device is to allow the players to play $\eta$, modeling the correlating device as private recommendations is without loss of generality.[8] When $\eta$ is a pure or uncorrelated mixed outcome, the correlation device can be dispensed with. To avoid extra notations we shall assume that all signals are associated with distinct posterior beliefs. All results go through without this assumption, although some may have to be rephrased to allow for the possibility of redundant signals.

**Assumption 1.** For each $i \in N$ and $a \in A$, there do not exist distinct elements $y_i, y_i' \in Y_i$ such that $p(y_{-i}|a, y_i) = p(y_{-i}|a, y_i')$ for all $y_{-i} \in Y_{-i}$.

We impose no further restriction on the correlation structure beyond Assumption 1. In general, the players' signals may be correlated and $p(\cdot|a)$ may not have full support. Hence, our model includes public monitoring as a special case.[9]

## 2.2 $T$-period contracting problem

In period 0, a principal proposes a contract. After observing the contract, the players play the stage game for $T$ periods. At the end of period $T$, the players report the private signals observed, and the correlating device reports the recommendations made during the $T$ periods. In addition to the stage-game payoffs, at the end of the $T$-period game, each player receives a transfer, stipulated by the contract, that depends on the reports of the players and the correlating device. While the correlating device always reports honestly, players may lie.

For each variable $x$, we use $x(t)$ to denote the period-$t$ value of $x$ and $x^t = (x(1), \ldots, x(t))$ to denote the history of $x$ up to period $t$. Hence, $\tilde{a}^T = (\tilde{a}(1), \ldots, \tilde{a}(T))$ is the history of recommendations and the report of the correlating device. Let $\hat{y}_i^T = (\hat{y}_i(1), \ldots, \hat{y}_i(T))$

[8]As is well known, the correlating device can be replaced by pregame communication when there are more than five players (Gerardi, 2004).

[9]The game becomes one of public monitoring when $Y_1 = \cdots = Y_n$ and for all $a \in A$, $p(y|a) > 0$ only if $y_1 = \cdots = y_n$.

denote the $T$-period signal-report of player $i$ and $\widehat{y}^T = (\widehat{y}_1^T, \ldots, \widehat{y}_n^T)$ denote the signal-report profile. A $T$-period contract consists of $n$ functions $w^T = (w_1^T, \ldots, w_n^T)$, where each $w_i^T$ maps each $(\widetilde{a}^T, \widehat{y}^T) \in A^T \times Y^T$ into a transfer such that the total transfer is weakly negative; i.e.,

$$\sum_{i=1}^{n} w_i^T(\widetilde{a}^T, \widehat{y}^T) \leq 0, \quad \text{for all } (\widetilde{a}^T, \widehat{y}^T) \in A^T \times Y^T.$$

Player $i$'s total discounted payoff is

$$\frac{1-\delta}{1-\delta^T}\left(\sum_{t=1}^{T} \delta^{t-1} g_i(a(t)) + w_i^T(\widetilde{a}^T, \widehat{y}^T)\right),$$

where $\delta \in (0,1)$ is a common discount factor for the players. The restriction to negative total transfer arises naturally in different contexts. For example, if bonus contracts are not legally enforceable, then the principal may have to commit to "burn" the difference between a lump sum and the actual bonus (MacLeod, 2003; Fuchs, 2007). In repeated games, players can enforce cooperation only by switching to inefficient continuation paths.

Since $N$, $A$, and $g$ are fixed in our analysis, we denote the $T$-period game by $\Gamma(\eta, T, \delta, w^T)$. A pure strategy of player $i$ consists of two components: an action strategy $\alpha_i^T$ that maps each $\left(\widetilde{a}_i^t, a_i^{t-1}, y_i^{t-1}\right) \in \bigcup_{s=1}^{T}\left(A_i^s \times A_i^{s-1} \times Y_i^{s-1}\right)$ into an action $a_i \in A_i$ and a reporting strategy $\rho_i^T$ that maps each $\left(\widetilde{a}_i^T, a_i^T, y_i^T\right) \in A_i^T \times A_i^T \times Y_i^T$ into a report $\widehat{y}_i^T \in Y_i^T$.[10] A mixed strategy $\sigma_i^T$ is a probability distribution over the set of pure strategies $(\alpha_i^T, \rho_i^T)$. Let $\Sigma_i^T$ denote the set of mixed strategies for player $i$.

Player $i$'s expected payoff conditional on $\sigma^T = \left(\sigma_1^T, \ldots, \sigma_n^T\right)$ is

$$v_i^T\left(\sigma^T; w_i^T\right) \equiv \frac{1-\delta}{1-\delta^T} E\left[\sum_{t=1}^{T} \delta^{t-1} g_i(a(t)) + w_i^T(\widetilde{a}^T, \widehat{y}^T) \,\middle|\, \sigma^T\right],$$

where the expectation is taken over $\left(\widetilde{a}^T, a^T, y^T, \widehat{y}^T\right)$ with respect to the distribution induced by $\sigma^T$, $\eta$, and $p$.

[10] As usual, $a^0$ denotes the null history $\emptyset$ and $A^0$ denotes the set whose only element is $a^0$. Similar notations apply for signal.

The contracting problem is to choose $w^T$ to enforce the correlated outcome $\eta$ throughout the contract. By the revelation principle, we can focus on mechanisms where players play the obedient strategies that follow recommendations in every period and report signals truthfully. Let $\sigma_i^{T*} = \left(\alpha_i^{T*}, \rho_i^{T*}\right)$ denote the obedient strategy of player $i$ and $\sigma^{T*} = \left(\sigma_1^{T*}, \ldots, \sigma_n^{T*}\right)$.

**Definition 1.** A contract $w^T$ enforces $\eta$ for $T$ periods if $\sigma^{T*}$ is a Nash equilibrium in $\Gamma(\eta, T, \delta, w^T)$. That is, if for all $i \in N$ and $\sigma_i^T \in \Sigma_i^T$,

$$v_i^T\left(\sigma^{T*}; w_i^T\right) \geq v_i^T\left(\sigma_i^T, \sigma_{-i}^{T*}; w_i^T\right).$$

An outcome $\eta$ is enforceable if it can be enforced by some $w^T$.

Obviously, if $\eta$ cannot be enforced when $T = 1$, then it cannot be enforced when $T > 1$. Conversely, if $\eta$ can be enforced when $T = 1$ by $w$, then it can be enforced for any $T$ by applying $w$ period by period. Thus, it is sufficient consider the case $T = 1$ to determine the enforceability of $\eta$.

In the following we write $\sigma$ for $\sigma^1$ and $w$ for $w^1$ for convenience. Let $\mu$ denote the distribution over $(\widetilde{a}, y)$ induced by $\eta$ and $p$. For all $(\widetilde{a}, y) \in A \times Y$,

$$\mu(\widetilde{a}, y) = p\left(y|\widetilde{a}\right)\eta\left(\widetilde{a}\right).$$

With a slight abuse of notation, we also use $\mu$ to denote the distribution of $(\widetilde{a}, \widehat{y})$ induced by the obedient strategy profile $\sigma^*$. Let $\pi^{\sigma_i}$ denote the distribution of $(\widetilde{a}, \widehat{y})$ when player $i$ deviates to $\sigma_i$, while other players choose $\sigma_{-i}^*$. For any $(\widetilde{a}, \widehat{y}) \in A \times Y$,

$$\pi^{\sigma_i}(\widetilde{a}, \widehat{y}) = \sum_{(\alpha_i, \rho_i)} \sigma_i(\alpha_i, \rho_i) \sum_{y_i : \rho_i(\widetilde{a}_i, \alpha_i(\widetilde{a}_i), y_i) = \widehat{y}_i} p\left(y_i, \widehat{y}_{-i} | \widetilde{a}_{-i}, \alpha_i\left(\widetilde{a}_i\right)\right)\eta\left(\widetilde{a}\right).$$

**Definition 2.** A deviating strategy $\sigma_i$ is *undetectable* if $\pi^{\sigma_i} = \mu$.

The following result due to Rahman (2012) provides a necessary and sufficient condition for enforceability.

**Lemma 1** (Theorem 1, Rahman, 2012)**.** *An action profile $\eta$ is enforceable if and only if for all $i \in N$ and all undetectable $\sigma_i$,*

$$\sum_{(\alpha_i, \rho_i)} \sigma_i(\alpha_i, \rho_i) \sum_{\widetilde{a} \in A} g_i\left(\alpha_i(\widetilde{a}_i), \widetilde{a}_{-i}\right)\eta\left(\widetilde{a}\right) \leq \sum_{\widetilde{a} \in A} g_i(\widetilde{a})\eta(\widetilde{a}).$$

Because the total transfer must be negative, enforcing a non-stage-game Nash equilibrium may come with a cost. The per-period efficiency loss of enforcing $\eta$ with $w^T$ in $\Gamma(\eta, T, \delta, w^T)$ is

$$W\left(\eta, T, \delta, w^T\right) \equiv -\sum_{i=1}^{n} \frac{1-\delta}{1-\delta^T} E\left[w_i(\widetilde{a}^T, \widehat{y}^T) | \sigma^{T*}\right].$$

Let $\mathscr{W}(\eta, T, \delta)$ be the set of $w^T$ that enforces $\eta$. The minimum per-period efficiency loss to enforce $\eta$ is

$$W^*(\eta, T, \delta) = \min_{w^T \in \mathscr{W}(\eta, T, \delta)} W\left(\eta, T, \delta, w^T\right).$$

Our objective is to characterize $W^*(\eta, T, \delta)$ as $T$ goes to infinity and $\delta$ goes to 1.

Before we proceed, a comment about the solution concept is in order. As is well known, Nash equilibrium imposes no restriction on players' responses off the equilibrium path. In our model, it is consistent with Nash equilibrium for players who observe signals inconsistent with the equilibrium actions to report honestly. Theorem 1, which establishes a lower bound on efficiency loss, continues to hold if the stronger notion of sequential equilibrium is used instead. Following Kandori and Matsushima (1998), Theorem 2, which establishes the tightness of the bound, can be made consistent with sequential equilibrium by assuming that the support of the signal distribution is invariant with $a$. Extending the result without invariant support would require specifying and keeping track of the players' diverging beliefs (as well as their beliefs about other players' continuation strategies) after one or multiple players observe inconsistent signals. We do not pursue this issue in this paper.

# 3 Motivating Example

It is useful to first revisit the argument of Abreu, Milgrom, and Pearce (1991). Consider a noisy Prisoners' Dilemma game. In each period, each player chooses $C$ or $D$ and then observes a public signal $y \in \{H, L\}$. The stage-game expected payoffs and the signal distributions under two action profiles $(C,C)$ and $(C,D)$ are given in Table 1. The signal distribution under $(D,C)$ is the same as that under $(C,D)$.

| | $C$ | $D$ |
|---|---|---|
| $C$ | $1,1$ | $-h,1+d$ |
| $D$ | $1+d,-h$ | $0,0$ |

Payoff matrix

| | $H$ | $L$ |
|---|---|---|
| $(C,C)$: | $p$ | $1-p$ |
| $(C,D)$: | $q$ | $1-q$ |

Signal distribution

Table 1: A prisoners' dilemma game with public signal.

If both players play $C$, then the signal is $H$ with probability $p$, and each player receives an expected payoff of 1. If one player chooses $C$ and the other chooses $D$, then the signal is $H$ with probability $q$, and the player who plays $D$ receives $(1+d)$, while the player who plays $C$ receives $-h$. Assume that $p > q$ and $d$, $h > 0$. The objective of the principal is to enforce $(C,C)$ in every period. Since the model is symmetric, we consider symmetric contracts.

Suppose the game is played once. To motivate the players to play $(C,C)$, each player $i$'s transfer when $y = H$ must be higher than the transfer when $y = L$. In particular, $w_i(H)$ and $w_i(L)$ need to satisfy the incentive-compatibility constraint

$$(p-q)\,(w_i(H) - w_i(L)) \geq d.$$

Given the constraint $w_i(H), w_i(L) \leq 0$, the most efficient way to enforce $(C,C)$ is to set

$$\begin{aligned} w_i(H) &= 0 \\ w_i(L) &= -\frac{d}{p-q}. \end{aligned}$$

The per-player efficiency loss is $(1-p)d/(p-q)$; see Figure 1.

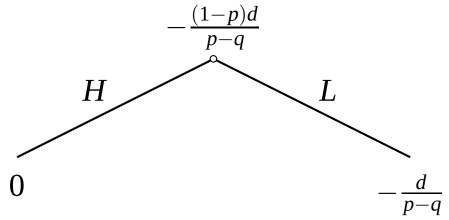


Figure 1: The one-period contract and efficiency loss.

Next, suppose the game is played twice with $\delta = 1$, but the first-period signal is delayed so that the players observe both the first- and second-period signals at the end of the second period; see the right panel of Figure 2.

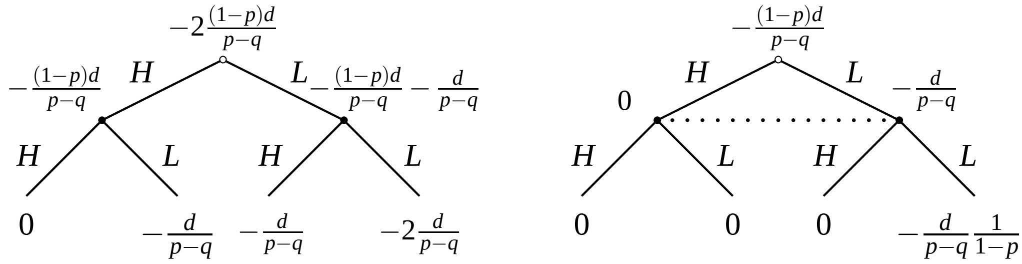


Figure 2: The two-period case.

Each player has four action strategies. Let $x(t)$ denote choosing action $x = C, D$ in period $t = 1, 2$. For example, $(D(1), C(2))$ represents the strategy that chooses $D$ in period 1 and $C$ in period 2. There are two binding incentive constraints:

$$(p-q)\left[p\left(w_i^2(H,H) - w_i^2(L,H)\right) + (1-p)\left(w_i^2(H,L) - w_i^2(L,L)\right)\right] \geq d \quad (1)$$
$$(p-q)\left[p\left(w_i^2(H,H) - w_i^2(H,L)\right) + (1-p)\left(w_i^2(L,H) - w_i^2(L,L)\right)\right] \geq d. \quad (2)$$

The first constraint requires that a player prefer $(C(1), C(2))$ to $(D(1), C(2))$; the second requires that he prefer $(C(1), C(2))$ to $(C(1), D(2))$.[11] One way to enforce $(C, C)$ twice is to apply the optimal short-term contract to each period separately, which is amount to setting

$$\begin{aligned} w_i^2(H,H) &= 0 \\ w_i^2(H,L) = w_i^2(L,H) &= -\frac{d}{p-q} \\ w_i^2(L,L) &= -\frac{2d}{p-q}. \end{aligned}$$

The per-player efficiency loss is $2(1-p)d/(p-q)$, twice the single-period loss. The contract, however, is not optimal. Minimizing the efficiency loss subject to (1) and (2)

[11] It turns out that the constraint that $(C(1), C(2))$ is preferable to $(D(1), D(2))$ is non-binding.

yields

$$\begin{aligned} w_i^2(H,H) = w_i^2(H,L) = w_i^2(L,H) &= 0 \\ w_i^2(L,L) &= -\frac{d}{(1-p)(p-q)}. \end{aligned}$$

The per-player efficiency loss remains $(1-p)d/(p-q)$, the single-period loss. Incentives are linked across periods under the optimal contract—the reward for the second-period $H$ (relative to $L$) depends on whether the first-period signal is $H$ or $L$. Linking reduces efficiency loss because punishing the players after $(H,L)$, as in the stationary contract above, makes it more costly to induce the players to choose $C$ in the first period. Abreu, Milgrom, and Pearce (1991) show that, if the game is repeated $T$ periods and all signals are observed at the end of period $T$, then, as $T$ becomes large, the per-period efficiency loss converges to 0.

The value of linking is limited by the public information that the players observe during the contract. Suppose the game is still played twice with $\delta = 1$, but the players now observe the first-period signal before choosing their second-period actions. The optimal contract in the delayed-signal case no longer works, as the players would have no incentive to choose $C$ in the second period after observing that the first-period signal is $H$. Since the signal is public, separate incentives must be provided to induce $(C,C)$ in the second period after different realizations of the first-period signal. Formally, three incentive-compatibility constraints must be satisfied; namely, the first period, the second period after $y(1) = H$, and the second period after $y(1) = L$. See the left panel of Figure 2. Let $w^2$ be a contract that enforces $(C,C)$ for two periods. The expected transfer conditional on the first-period signal $y(1)$ and second-period action profile $(C,C)$ is $E\left[w_i^2|y(1),CC\right]$. Since enforcing $(C,C)$ in the second period after $y(1) = H$ is the same as enforcing $(C,C)$ in a single-period game,

$$E\left[w_i^2|H,CC\right] \leq -(1-p)\frac{d}{p-q}. \tag{3}$$

In addition, the incentive-compatibility constraint in period 1 requires that

$$(p-q)\left(E\left[w_i^2|H,CC\right] - E\left[w_i^2|L,CC\right]\right) \geq d. \tag{4}$$

Combining (3) and (4), we have

$$E\left[w_i^2|L,CC\right] \leq -(2-p)\frac{d}{p-q}.$$

It follows that

$$pE\left[w_i^2|H,CC\right]+(1-p)E\left[w_i^2|L,CC\right]\leq -2(1-p)\frac{d}{p-q}.$$

The expected loss for enforcing $(C,C)$ for two periods is therefore twice the efficiency loss for enforcing $(C,C)$ for one period. To induce $(C,C)$ in the first period, the expected transfer after the first-period $L$ signal must be lower than the expected transfer after the first-period $H$ signal. Since player $i$'s expected transfer after the first-period $H$ signal is lowered to provide incentives in the second period, the expected transfer after the first-period $L$ signal must be lowered correspondingly. Unlike the delayed-signal case, here linking incentives across periods does not reduce efficiency loss. More generally, Abreu, Milgrom, and Pearce (1991) show that the per-player efficiency loss for enforcing $(C,C)$ for $T$ periods is $T$ times the single-period loss when each period's signal is observed immediately at the end of the period.

# 4 Self-Evident Events

In the example in Section 3, what is special about a public signal is not merely that it is observed by everyone, but also that everyone knows that it is observed by everyone, *ad infinitum*. Only then are the continuation games after $H$ and after $L$ entirely separated.

In our model, players do not directly observe the recommendations and signals of the other players. Instead, they form beliefs conditional on their own recommendation and signal. Conditional on $\eta$, the recommendation and signal pair $(\tilde{a},y)$ is distributed according to $\mu$. Let $P_i$ denote player $i$'s information partition of $\text{supp}(\mu)\subset A\times Y$. The partition element of $P_i$ that contains $(\tilde{a},y)$ is denoted by $P_i(\tilde{a},y)$. For any $i\in N$ and any $(\tilde{a},y),(\tilde{a}',y')\in\text{supp}(\mu)$,

$$\left(\tilde{a}',y'\right)\in P_i\left(\tilde{a},y\right)\ \text{ if and only if }\ \left(\tilde{a}_i',y_i'\right)=\left(\tilde{a}_i,y_i\right).$$

In the terminology of interactive epistemology, a subset $E$ of $\text{supp}(\mu)$ is an event. Player $i$ "knows" that $E$ occurs given his belief at $(\tilde{a},y)$ if $P_i(\tilde{a},y)\supseteq E$. An event $E$ is common belief among the players at $(\tilde{a},y)$ if, when $(\tilde{a},y)$ occurs, every player knows that $E$ occurs, knows that everyone knows that $E$ occurs, and so on. An event $E$ is self-evident if it is common belief at every $(\tilde{a},y)\in E$. A self-evident event is irreducible if it contains no proper subset that is self-evident. Let $P$ denote the meet of $(P_1,\ldots,P_n)$ (i.e., the least common coarsening). Any element of $P$ is self-evident and irreducible (Chapter 5 of Osborne and Rubinstein, 1994). When $\eta$ is pure and the signal is public, every realization of the public signal is self-evident. When $\eta$ is pure and $p$ has full support, the only self-evident event is the support of $\mu$. In general, there may be multiple irreducible self-evident events, each containing multiple signal profiles.

| | $h_2$ | $m_2$ | $l_2$ |
|---|---|---|---|
| $h_1$ | 0.5 | 0 | 0 |
| $m_1$ | 0 | 0.1 | 0 |
| $l_1$ | 0 | 0.2 | 0.2 |

Signal distribution

| | $h_2$ | $m_2$ | $l_2$ |
|---|---|---|---|
| $h_1$ | $-1$ | 0 | 0 |
| $m_1$ | 0 | 0 | 0 |
| $l_1$ | 0 | $-2$ | $-3$ |

Symmetric contract $w_1 = w_2$

Table 2: Enforcing a pure action profile in a two-player game.

For example, the left panel of Table 2 describes the signal distribution of a two-player game under a pure action profile $\widetilde{a}$. In this case, the meet contains two elements:

$$\{(\widetilde{a},h_1,h_2)\},\quad \{(\widetilde{a},m_1,m_2),(\widetilde{a},l_1,m_2),(\widetilde{a},l_1,l_2)\}.$$

The notion of self-evident events captures what becomes "public" during the game. But unlike public signals, which is public both on and off the equilibrium path, a self-evident event is self-evident only on the equilibrium path. Let $\omega \in P$ denote a self-evident event and $E\left[w_i(\widetilde{a},\widehat{y})|\sigma^*,\omega\right]$ denote player $i$'s expected transfer conditional on $\sigma^*$ and $\omega$. Fix some $\omega^* \in \arg\max_{\omega\in P}\sum_{i=1}^{n} E\left[w_i(\widetilde{a},\widehat{y})|\sigma^*,\omega\right]$. We can write

$$w_i(\widetilde{a},\widehat{y}) = w_{i,a}(\widetilde{a},\widehat{y}) + w_{i,b}(\widetilde{a},\widehat{y}),$$

where

$$\begin{aligned} w_{i,a}(\widetilde{a},\widehat{y}) &\equiv w_i(\widetilde{a},\widehat{y}) - w_{i,b}(\widetilde{a},\widehat{y}) \\ w_{i,b}(\widetilde{a},\widehat{y}) &\equiv E\left[w_i\left(\widetilde{a}',\widehat{y}'\right)|\sigma^*,P(\widetilde{a},\widehat{y})\right] - E\left[w_i\left(\widetilde{a}',\widehat{y}'\right)|\sigma^*,\omega^*\right]. \end{aligned}$$

The decomposition divides $w_i$ into a self-evident component, $w_{i,b}$, which depends solely on $P(\widetilde{a},\widehat{y})$, and a residual private component, $w_{i,a}$.

By construction, for all $(\widetilde{a},\widehat{y})$,

$$\sum_{i=1}^{n} w_{i,b}(\widetilde{a},\widehat{y}) \leq 0,$$

and, for all $i \in N$ and all $\omega \in P$,

$$E\left[w_{i,a}(\widetilde{a},\widehat{y})|\sigma^*,\omega\right] = E\left[w_i\left(\widetilde{a},\widehat{y}\right)|\sigma^*,\omega^*\right].$$

The efficiency loss due to the self-evident component is

$$\begin{aligned} L(\eta,w) &\equiv -\sum_{i=1}^{n} E\left[w_{i,b}\left(\widetilde{a},\widehat{y}\right)|\sigma^*\right] && (5) \\ &= W(\eta,w) + \sum_{i=1}^{n} E\left[w_i\left(\widetilde{a},\widehat{y}\right)|\sigma^*,\omega^*\right]. && (6) \end{aligned}$$

Returning to the example in Table 2, the total conditional transfers for the symmetric contract in the right panel of Table 2 are $-2$ and $-4$, respectively, and, thus,

$$L(\eta,w) = \frac{2+4}{2} + \max(-2,-4) = 1.$$

Let

$$L^*(\eta) \equiv \min_{w \in \mathscr{W}(\eta)} L(\eta,w) \tag{7}$$

denote the minimum efficiency loss of the self-evident component among contracts that enforce $\eta$. When $\eta$ is pure and the signal is public, the second term in (6) is zero and hence

$$L^*(\eta) = \min_{w \in \mathscr{W}(\eta)} W(\eta,w). \tag{8}$$

When $\eta$ is pure and $p$ has full support, $L^*(\eta) = 0$.

Theorem 1 says that the per-period efficiency loss is bounded from below by the efficiency loss of the self-evident component. It immediately implies that linking has no value when $\eta$ is pure and the signal is public.

**Theorem 1.** *For any enforceable* $\eta$, $W^*(\eta,T,\delta) \geq L^*(\eta)$ *for any* $T \geq 1$ *and* $\delta \leq 1$.

We saw in the example in Section 3 that the need to provide separate incentives in the continuation games after different public events limits the value of linking. Theorem

1 extends this idea. Consider the two-period case with $\delta = 1$. Suppose the two-period contract $w^2$ enforces $\eta$. Let $\overline{w}(\widetilde{a}(1), y(1))$ denote the expected value of $w^2$ conditional on the first-period outcome $(\widetilde{a}(1), y(1))$, assuming that $\eta$ will be played in the second period. Let $E[\overline{w}_i(\widetilde{a}(1), y(1))|\mu]$ denote the expected value of $\overline{w}$ when $(\widetilde{a}(1), y(1))$ is distributed according to $\mu$. We can write

$$\begin{aligned}\sum_{i=1}^{n} E\left[w_i^2\left(\widetilde{a}^2, y^2\right)|\sigma^{2*}\right] &= \sum_{i=1}^{n} E\left[\overline{w}_i(\widetilde{a}(1), y(1))|\mu\right] \\ &= \left[\sum_{i=1}^{n} E\left[\overline{w}_i(\widetilde{a}(1), y(1))|\mu\right] - \sum_{i=1}^{n} E\left[\overline{w}_i(\widetilde{a}(1), y(1))|\mu, \omega^*\right]\right] \\ &\quad + \sum_{i=1}^{n} E\left[\overline{w}_i(\widetilde{a}(1), y(1))|\mu, \omega^*\right].\end{aligned}$$

Note that $\overline{w}$ can also be taken as a short-term contract that pays player $i$ the amount $\overline{w}_i(\widetilde{a}(1), y(1))$ when the correlating device and players report $(\widetilde{a}, \widehat{y}) = (\widetilde{a}(1), y(1))$. Since $w^2$ enforces $\eta$ in $\Gamma(\eta, 2, 1, w^2)$, $\overline{w}$ must enforce $\eta$ in the stage game. Hence, the term in the square bracket on the right-hand side of the second equality must be less than $-L^*(\eta)$. Furthermore, since $\omega \in P$ is self-evident, the players' conditional beliefs at any $(\widetilde{a}, y) \in \omega$ can be derived from a common prior $\mu(\cdot|\omega)$ through Bayes' rule. We can, therefore, take the continuation game after each $\omega \in P$ as a stage game endowed with an extra correlating device that draws $(\widetilde{a}, \widehat{y})$ according to $\mu(\cdot|\omega)$ and informs each player $i$ of $(\widetilde{a}_i, \widehat{y}_i)$. Given this extra correlating device, the expected total transfer for inducing $\eta$ in the second period after $\omega$ is $\sum_{i=1}^{n} E\left[\overline{w}_i(\widetilde{a}(1), y(1))|\mu, \omega\right]$. Since the stage game is already endowed with a correlating device, having a second one does not reduce the second-period enforcement cost. Thus, for any $\omega \in P$,

$$\begin{aligned}\sum_{i=1}^{n} E\left[\widetilde{w}_i(\widetilde{a}(1), y(1))|\mu, \omega\right] &\leq -\max_{w \in \mathscr{W}(\eta)} W(w, \eta) \\ &\leq -L^*(\eta).\end{aligned}$$

The second inequality holds because $W(w, \eta) \geq L(w, \eta)$ for any $w \in \mathscr{W}(\eta)$. Since the square-bracket term is also less than $-L^*(\eta)$, $W(\eta, 2, 1, w^2)$ must be greater than $2L^*(\eta)$.

## 5 No Free Information

In this section, we show that the bound in Theorem 1 is tight under a condition that we call "no free information." Under a non-stationary contract that links incentives, players could benefit from obtaining information about previous outcomes. In the two-period noisy Prisoners' Dilemma in Section 3, under a contract that punishes only after two $L$ signals, a player is willing to pay to learn the first-period signal at the end of the first period. In Blackwell (1953), given a fixed distribution of states, one experiment is more informative than another if the latter can be expressed as a garbling of the former. In our model, we can also think of a player's action as an experiment to generate information about the actions and signals of the other players. Let $\eta_i$ denote the marginal distribution of player $i$'s action under $\eta$. Let $\gamma_i \in \Delta(A_i)$ denote a mixed action for player $i$, where $\gamma_i(a_i)$ is the probability of choosing $a_i$.

**Definition 3.** For any $\gamma_i, \gamma_i' \in \Delta(A_i)$, $\gamma_i$ is more informative than $\gamma_i'$ at the recommendation $\widetilde{a}_i \in$supp$(\eta_i)$ if for any $(a_i, y_i) \in A_i \times Y_i$, there exists a distribution $\lambda_{(a_i,y_i)}(\cdot,\cdot) \in \Delta(A_i \times Y_i)$ such that for all $(\widetilde{a}_{-i}, y_{-i}) \in A_{-i} \times Y_{-i}$ and all $(a_i', y_i') \in A_i \times Y_i$,

$$\sum_{(a_i,y_i)\in A_i\times Y_i} \lambda_{(a_i,y_i)}\left(a_i', y_i'\right)\gamma(a_i)\, p\left(y_{-i}, y_i|\widetilde{a}_{-i}, a_i\right)\eta(\widetilde{a}) = \gamma'\left(a_i'\right) p\left(y_{-i}, y_i'|\widetilde{a}_{-i}, a_i'\right)\eta(\widetilde{a}). \tag{9}$$

A mixed action $\gamma_i$ is strictly more informative than $\gamma_i'$ if $\gamma_i$ is more informative than $\gamma_i'$ but not vice versa.

Unlike the standard set-up where the distribution of states is fixed, in our model player $i$'s action may alter the distribution of $y_{-i}$. Equation (9) requires that for every $\widetilde{a}_{-i}$ with $\eta(\widetilde{a}_i, \widetilde{a}_{-i}) > 0$ (assuming that the other players are following the recommendations) $\gamma_i$ lead to the same distribution of $y_{-i}$ that $\gamma_i'$ induces, and be more informative than $\gamma_i'$ in the Blackwell sense. Since $\{\lambda_{(a_i,y_i)}(\cdot)|(a_i,y_i) \in A_i \times Y_i\}$ can be interpreted as a mixed reporting strategy, an equivalent definition is to say that $\gamma_i$ is more informative than $\gamma_i'$ if player $i$ can choose $\gamma_i$ and misreport $(a_i, y_i)$ to mimic the distribution of $(a_i, y)$ under $\gamma_i'$.

**Definition 4.** An action profile $\eta$ satisfies the no-free-information condition if

$$\sum_{a_i\in A_i}\gamma_i(a_i)\sum_{\widetilde{a}_{-i}\in A_{-i}} g(a_i,\widetilde{a}_{-i})\eta(\widetilde{a}) < \sum_{\widetilde{a}_{-i}\in A_{-i}} g(\widetilde{a})\,\eta(\widetilde{a})$$

for any $i \in N$, $\widetilde{a}_i \in \operatorname{supp}(\eta_i)$, and $\gamma_i$ strictly more informative than $\widetilde{a}_i$ at $\widetilde{a}_i$.

In words, $\eta$ satisfies the no-free-information condition if any deviation that generates more information for a player must strictly lower his stage-game payoff.

**Theorem 2.** *If $\eta$ is enforceable and satisfies the no-free-information condition, then for any $\varepsilon > 0$, there exists $T_0$ such that, for any $T \geq T_0$ and any $\delta \geq 1 - T^{-2}$, $W^*(\eta, T, \delta) \leq L^*(\eta) + \varepsilon$.*

Theorem 2 says that so long as there is a cost to obtain extra information, $\eta$ can be enforced with per-period efficiency loss arbitrarily close to $L^*(\eta)$, when the players are sufficiently patient and the contract is sufficiently long. Since $L^*(\eta) = 0$ when $\eta$ is pure and $P$ is a singleton, it follows immediately that a pure action profile $a$ can be enforced efficiently in the long run when $p(\cdot|a)$ has full support.

## 5.1 Almost strict enforceability

The proof of Theorem 2 is more complex than that of the two-period noisy Prisoners' Dilemma example in Section 3. Since the players in the example do not observe any signals until the end of the contract, the principal has complete freedom in linking across periods. Here, the incentives following different self-evident events must be provided separately. Furthermore, within each irreducible self-evident event, players who observe different private signals face different incentive-compatibility constraints. Due to these complications, it would be difficult to derive the optimal contract explicitly as in Section 3.

Our approach is to mimic a sequence of identical short-term contracts, under which players will have no incentive to acquire information about past outcomes, with a more efficient non-stationary long-term contract. For this approach to work, we need the initial short-term contract to be robust so that a long-term contract "close" to it will also enforce $\eta$. We show that the no-free-information condition ensures the existence of such a robust short-term contract. Previous results in the private-monitoring literature (e.g., Compte, 1998 and Obara, 2009) assume that the equilibrium outcome is strictly enforceable. Write $v_i$ for $v_i^1$, the total payoff under a short-term contract.

**Definition 5** (Strict enforceability)**.** A contract $w$ strictly enforces $\eta$ if, for any player $i$ and any strategy $\sigma_i \in \Sigma_i$,

$$v_i(\sigma^*; w_i) \geq v_i\left(\sigma_i, \sigma^*_{-i}; w_i\right),$$

with the inequality strict for any $\sigma_i$ that involves player $i$'s not following the mediator's recommendation with strictly positive probability. An action profile is strictly enforceable if it can be enforced strictly by some $w$.

If $w$ strictly enforces $\eta$, then under $w$ a player is strictly worse off whenever he does not follow the recommendation. We can define a weaker form of strict enforceability.

**Definition 6** (Almost-strict enforceability)**.** A contract $w$ almost strictly enforces $\eta$ if, for any player $i$ and any strategy $\sigma_i \in \Sigma_i$,

$$v_i(\sigma^*; w_i) \geq v_i\left(\sigma_i, \sigma^*_{-i}; w_i\right),$$

with the inequality strict for any detectable $\sigma_i$. An action profile is almost-strictly enforceable if it can be enforced almost strictly by some $w$.

Unlike strict enforceability, almost-strict enforceability requires only that a player be strictly worse off when the deviation is detectable. The no-free-information condition is between these two notions of enforceability.

**Lemma 2.** *A strictly enforceable action profile satisfies the no-free-information condition. An enforceable action profile that satisfies the no-free-information condition is almost-strictly enforceable.*

The first part of the lemma follows because strict enforceability implies that any deviation from a recommended action, detectable or not, must yield a strictly lower stage-game payoff. For the second part, if $\eta$ is not almost strictly enforceable, then, by the theory of alternatives, there exists an undetectable mixed strategy $\sigma_i$ with a support of detectable strategies that yields a higher stage-game payoff for some player $i$, violating the no-free-information condition.

The converse of Lemma 2 is false. No free information, together with enforceability, is weaker than strict enforceability, as deviations from recommended actions may be undetectable but not more informative. Almost-strict enforceability does not rule out pure undetectable deviations more informative than the obedient strategy. Theorem 2 implies that strict enforceability is not necessary to attain the bound in Theorem 1. While we do not have a proof, we speculate that Theorem 2 does not hold when the no-free-information condition is replaced by almost-strict enforceability.[12]

## 5.2 Linking incentives

In this section, we describe the construction of the long-term contract. For any $\varepsilon > 0$, let $w^*$ denote a contract that enforces $\eta$ almost strictly with

$$L(\eta, w^*) \leq L^*(\eta) + \frac{\varepsilon}{2}. \tag{10}$$

Note that if both $w'$ and $w''$ enforce $\eta$, the latter almost strictly, then any convex combination of the two that has a strictly positive weight on the latter enforces $\eta$ almost strictly. Hence, there exists $w^*$ that satisfies (10) for any $\varepsilon > 0$.

Let $w^{T*}$ denote the $T$-period version of $w^*$. For all $i \in N$ and all $(\widetilde{a}^T, \widehat{y}^T)$,

$$w_i^{T*}(\widetilde{a}^T, \widehat{y}^T) = \sum_{t=1}^{T} \delta^{t-1} w_i^* (\widetilde{a}(t), \widehat{y}(t)).$$

By construction, under $w_i^{T*}$ player $i$ is strictly worse off if he chooses a deviating strategy that induces a distribution of $(\widetilde{a}, \widehat{y})$ that is different from $\mu$ in any period $t \leq T$.

Let

$$\begin{aligned} w_{i,b}^*(\widetilde{a}, \widehat{y}) &= E\left[w_i^*\left(\widetilde{a}', \widehat{y}'\right) | \sigma^*, P(\widetilde{a}, \widehat{y})\right] - E\left[w_i^*\left(\widetilde{a}', \widehat{y}'\right) | \sigma^*, \omega^*\right] \\ w_{i,a}^*(\widetilde{a}, \widehat{y}) &= w_i^*(\widetilde{a}, \widehat{y}) - w_{i,b}^*(\widetilde{a}, \widehat{y}) \end{aligned}$$

denote, respectively, the self-evident and residual private components of $w_i^*$ (as defined in Section 4). We shall construct a new contract under which each player effectively is paid $w_{i,b}^*$ and the part of $w_{i,a}^*$ that is below the mean. The following lemma is crucial for the construction.

[12] Our proof of Theorem 2 does not hold if no free information is replaced by almost-strict enforceability.

**Lemma 3.** *For any $i \in N$, there exists a function $z_{i,j} : A \times Y \to \mathbf{R}$ for each $j \in N$ satisfying the following conditions:*

*(i)* $(n-1)z_{i,i}(\widetilde{a},\widehat{y}) = \sum_{j \neq i} z_{i,j}(\widetilde{a},\widehat{y})$ *for all* $(\widetilde{a},\widehat{y}) \in A \times Y$.

*(ii)* $E[z_{i,i}(\widetilde{a}',\widehat{y}')|\sigma^*,\widetilde{a}_i,y_i] = E[w^*_{i,a}(\widetilde{a}',\widehat{y}')|\sigma^*,\widetilde{a}_i,y_i]$ *for all* $(\widetilde{a}_i,y_i) \in A_i \times Y_i$.

*(iii)* $E[z_{i,j}(\widetilde{a}',\widehat{y}')|\sigma^*,\widetilde{a}_j,y_j] = E[w^*_{i,a}(\widetilde{a}',\widehat{y}')|\sigma^*,P(\widetilde{a},y)]$ *for all* $(\widetilde{a},y) \in A \times Y$.

Lemma 3, which critically relies on the irreducibility of each member of $P$, says that, for any $i \in N$, there is a variable whose expected value conditional on player $i$'s private information is the same as that of $w^*_{i,a}$ conditional on the same information.[13] Furthermore, the variable can be expressed as the average of $(n-1)$ variables, each of which has the same expectation as that of $w^*_{i,a}$ but is uncorrelated with the private information of a distinct player $j$.

Henceforth, consider a specific set of $z_{i,j}$, $i, j \in N$, that satisfies Lemma 3. For any $i, j \in N$, let

$$\begin{aligned} w^{T*}_{i,a}(\widetilde{a}^T,\widehat{y}^T) &= \sum_{t=1}^{T} \delta^{t-1} w^*_{i,a}(\widetilde{a}(t),\widehat{y}(t)) \\ w^{T*}_{i,b}(\widetilde{a}^T,\widehat{y}^T) &= \sum_{t=1}^{T} \delta^{t-1} w^*_{i,b}(\widetilde{a}(t),\widehat{y}(t)) \\ z^{T}_{i,j}(\widetilde{a}^T,\widehat{y}^T) &= \sum_{t=1}^{T} \delta^{t-1} z_{i,j}(\widetilde{a}(t),\widehat{y}(t)) \end{aligned}$$

denote the discounted sums of $w^*_{i,a}$, $w^*_{i,b}$, and $z_{i,j}$, respectively.

Fix some small $\kappa_T > 0$. Let

$$\begin{aligned} R^+_i(\widetilde{a}^T,\widehat{y}^T) &= \max\left(0, w^{T*}_{i,a}(\widetilde{a}^T,\widehat{y}^T) - E[w^{T*}_{i,a}(\widetilde{a}^{T\prime},\widehat{y}^{T\prime})|\sigma^{T*}] - \kappa_T\right) \\ R^-_i(\widetilde{a}^T,\widehat{y}^T) &= \min\left(0, w^{T*}_{i,a}(\widetilde{a}^T,\widehat{y}^T) - E[w^{T*}_{i,a}(\widetilde{a}^{T\prime},\widehat{y}^{T\prime})|\sigma^{T*}] - \kappa_T\right) \end{aligned}$$

denote, respectively, the positive and negative parts of the difference between $w^{T*}_{i,a}$ and the mean of $w^{T*}_{i,a}$ plus $\kappa_T$.

[13] While Lemma 3 formally applies to $w^*_{i,a}$, it holds for any function from $A \times Y$ to $\mathbf{R}$.

We can now formally define a new contract $w^{T**}$. For all $(\widetilde{a}^T, \widehat{y}^T)$, define

$$w_i^{T**}(\widetilde{a}^T, \widehat{y}^T) = R_i^-(\widetilde{a}^T, \widehat{y}^T) + w_{i,b}^{T*}\left(\widetilde{a}^T, \widehat{y}^T\right) + \left[R_i^+(\widetilde{a}^T, \widehat{y}^T) I_{i,i}(\widetilde{a}^T, \widehat{y}^T) - \sum_{j \neq i} R_j^+(\widetilde{a}^T, \widehat{y}^T) I_{j,i}(\widetilde{a}^T, \widehat{y}^T)\right], \quad (11)$$

where $I_{i,j}(\widetilde{a}^T, \widehat{y}^T)$ is an indicator function that equals 1 if

$$z_{i,j}^T(\widetilde{a}^T, \widehat{y}^T) > E\left[w_{i,a}^{T*}\left(\widetilde{a}^{T\prime}, \widehat{y}^{T\prime}\right) | \sigma^{T*}\right] + \frac{1}{2}\kappa_T.$$

Under this new contract, each player $i$ is paid the self-evident component of $w_i^{T**}$, the part of the residual private component that is less than the mean plus $\kappa_T$, and a third component (inside the square bracket) that pays player $i$ $R_i^+$ when $I_{i,i} = 1$ and $-R_j^+$ when $I_{j,i} = 1$.

Notice that both $w_i^{T*}$ and $w_i^{T**}$ pay $w_{i,b}^{T*}$ in full but, instead of paying $w_{i,a}^{T*}$ in full, $w_i^{T**}$ pays only the part of $w_{i,a}^{T*}$ below the mean plus $\kappa_T$. The truncation reduces efficiency loss because the mean (summing over all players) is negative.[14] While the truncation distorts incentives, the distortion is small ex ante because $w_{i,a}^{T*}$ is unlikely to be greater than the mean by $\kappa_T$ (even when $\kappa_T$ is small) when $T$ is large. Players may update their beliefs about the likelihood of the truncation during the contract. The last component in (11) is designed so that the total transfer is negative, and the distortion effect is small throughout the game and diminishes exponentially to zero as $T$ goes to infinity.

To see that the total transfer is negative, note that for any $(\widetilde{a}^T, \widehat{y}^T)$,

$$\sum_{i=1}^n w_i^{T**}(\widetilde{a}^T, \widehat{y}^T) = \sum_{i=1}^n R_i^-(\widetilde{a}^T, \widehat{y}^T) + \sum_{i=1}^n w_{i,b}^{T*}(\widetilde{a}^T, \widehat{y}^T) + \sum_{i=1}^n R_i^+(\widetilde{a}^T, \widehat{y}^T)\left(I_{i,i}(\widetilde{a}^T, \widehat{y}^T) - \sum_{j \neq i} I_{i,j}(\widetilde{a}^T, \widehat{y}^T)\right). \quad (12)$$

By definition, both the first and second summation terms on the right-hand side of (12) are negative. Since $(n-1)z_{i,i}^T$ is equal to the sum of $z_{i,j}^T$ over all $j \neq i$ (Lemma 3), if

[14] If we think of $w_{i,a}^{T*}$ as a punishment, the new contract punishes only when player $i$ performs worse than expectation.

$I_{i,i}(\widetilde{a}^T,\widehat{y}^T)=1$, then there must exist some $j\neq i$ such that $I_{i,j}(\widetilde{a}^T,\widehat{y}^T)=1$. Hence, the last summation term is also negative.

Rearranging the terms on the right-hand side of (11), we can write

$$w_i^{T**}(\widetilde{a}^T,\widehat{y}^T)=w_i^{T*}(\widetilde{a}^T,\widehat{y}^T)-E[w_{i,a}^{T*}(\widetilde{a}^{T\prime},\widehat{y}^{T\prime})|\sigma^{T*}]-\kappa_T-\phi_i(\widetilde{a}^T,\widehat{y}^T), \tag{13}$$

where

$$\phi_i(\widetilde{a}^T,\widehat{y}^T)=R_i^+(\widetilde{a}^T,\widehat{y}^T)(1-I_{i,i}(\widetilde{a}^T,\widehat{y}^T))+\sum_{j\neq i}R_j^+(\widetilde{a}^T,\widehat{y}^T)I_{j,i}(\widetilde{a}^T,\widehat{y}^T)$$

measures the distortion in incentives.

The per-period efficiency loss of $w^{T**}$ is

$$\begin{aligned}&-\frac{1-\delta}{1-\delta^T}\sum_{i=1}^n E\left[w_i^{T*}\left(\widetilde{a}^T,\widehat{y}^T\right)-E[w_{i,a}^{T*}(\widetilde{a}^{T\prime},\widehat{y}^{T\prime})|\sigma^{T*}]-\kappa_T-\phi_i(\widetilde{a}^T,\widehat{y}^T)|\sigma^{T*}\right]\\ =&L(\eta,w^*)+\frac{1-\delta}{1-\delta^T}\left(n\kappa_T+\sum_{i=1}^n E[\phi_i(\widetilde{a}^T,\widehat{y}^T)|\sigma^{T*}]\right).\end{aligned} \tag{14}$$

The per-period efficiency loss converges to $L(\eta,w^*)$ if the second term in the last equation converges to zero as $\delta$ goes to one and $T$ goes to infinity. Set $\kappa_T=T^{2/3}$ so that $\kappa_T/T$ converges to zero. Let $H_i^{T*}$ denote the set of histories that player $i$ may observe during the $T$-period contract under $\sigma^{T*}$. The following lemma shows that the expected value of $\phi_i(\widetilde{a}^T,\widehat{y}^T)$ conditional on any private information player $i$ may learn during the game on the equilibrium path diminishes uniformly and exponentially with $T$.

**Lemma 4.** *Let $\kappa_T=T^{2/3}$. There exists $c>0$ such that for all $i\in N$, $T\geq 1$, and $h_i\in H_i^{T*}$,*

$$E\left[\phi_i(\widetilde{a}^T,\widehat{y}^T)|\sigma^{T*},h_i\right]<ncT\exp\left(-\frac{T^{1/3}}{8c^2}\right).$$

Lemma 4 means that, when $T$ is sufficiently large, a player $i$ who has followed $\sigma_i^{T*}$ up to some period $t\leq T$ (conditional on any private information that he may observe from period 1 to $(t-1)$) will believe that if he follows the equilibrium strategy $\sigma_i^{T*}$ in the remaining periods, he will obtain a $\phi_i$ close to zero. This, together with the fact that $\phi_i$ is always positive, means that no deviation can reduce the expected value of $\phi_i$ significantly. Since $w_i^{T*}$ enforces $\eta$ almost strictly and $\eta$ satisfies the no-free-information condition, under $w_i^{T*}$ a player $i$ deviating in any period must be strictly worse off if the deviation is detectable or involves choosing an action more informative than the recommended one. Since the effect of a single-period deviation on the total payoff is of the order $1/T$ (as $\delta$ goes to one), the players will have the incentives to play $\eta$ under $w^{T**}$ when $T$ is sufficiently large, as the distortion in incentives due to the truncation diminishes at a rate faster than $1/T$.

We prove Lemma 4 by showing that the probability that $I_{i,i}(\widetilde{a}^T, \widehat{y}^T)$ is zero when $R_i^+(\widetilde{a}^T, \widehat{y}^T)$ is significantly bigger than zero and the probability that $I_{j,i}(\widetilde{a}^T, \widehat{y}^T)$ is one both diminish exponentially to zero as $T$ goes to infinity. Figure 3 illustrates the idea. The distribution on the left in the top panel is the ex ante distribution of $w_{i,a}^{T*}$. When

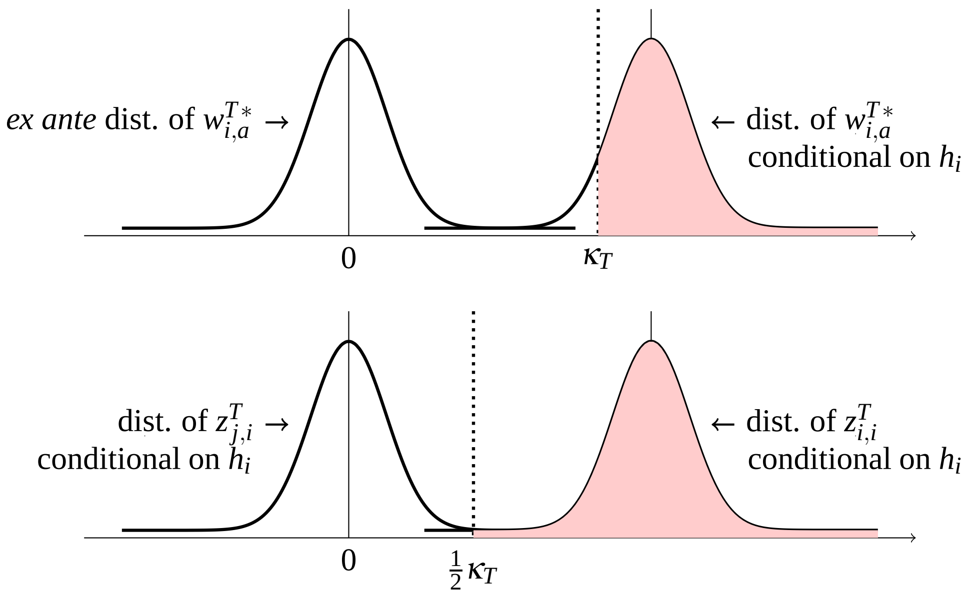


Figure 3: Distributions of $w_{i,a}^{T*}$, $z_{j,i}^T$ and $z_{i,i}^T$.

$T$ is large, $w_{i,a}^{T*}$ is almost always close to the mean (normalized to zero here). Hence, the probability that $w_{i,a}^{T*}$ exceeds $\kappa_T$ is very small ex ante. Nevertheless, the distribution may shift to the right conditional on some $h_i \in H_i^{T*}$; see the top panel of Figure 3. Note that the probability that $w_{i,a}^{T*}$ exceeds $\kappa_T$ conditional on $h_i$ is significantly bigger than zero (the shaded area). The lower panel of Figure 3 describes the distributions of $z_{i,i}^T$ and $z_{j,i}^T$ conditional on the same $h_i$. Since, by construction, $w_{i,a}^{T*}$ and $z_{i,i}^T$ have the same conditional mean, the probability that $z_{i,i}^T$ exceeds $\kappa_T/2$ must be very close to one (the shaded area in the lower panel). Hence, $I_{i,i}$ is unlikely to be zero when $R_i^+$ is significantly greater than zero. But $z_{j,i}^T$ is, by construction, uncorrelated with $h_i \in H_i^{T*}$. Hence, conditional on $h_i$ the probability that $z_{j,i}^T$ exceeds $\kappa_T/2$ remains very low.

# 6 Value of Linking

In this section, we characterize $L^*(\eta)$ in terms of the primitives of the model. The following theorem provides a sufficient condition for $L^*(\eta) = 0$.

**Theorem 3A.** For any enforceable $\eta$, we have $L^*(\eta) = 0$ if for any player $i \in N$, any deviating strategy $\sigma_i \in \Sigma_i / \{\sigma_i^*\}$ satisfies one of the following conditions:

(i) There exists $(\widetilde{a}, \widehat{y})$ such that $\pi^{\sigma_i}(\widetilde{a}, \widehat{y}) > 0$ and $\mu(\widetilde{a}, \widehat{y}) = 0$.

(ii) There exists $\omega \in P$ such that $\pi^{\sigma_i}(\cdot|\omega) \neq \mu(\cdot|\omega)$.

(iii) There exists a player $j \in N$ such that there is no $\sigma_j$ with $\pi^{\sigma_j} = \pi^{\sigma_i}$.

Theorem 3A identifies three types of deviations that can be deterred almost costlessly as $T$ becomes large. First, deviations that may result in $(\widetilde{a}, \widehat{y})$ outside of the support of $\mu$ can be deterred costlessly by a contract that punishes all players severely when an out-of-support $(\widetilde{a}, \widehat{y})$ occurs. Second, deviations that change the distribution of $(\widetilde{a}, \widehat{y})$ conditional on some $\omega \in P$ can be deterred by a contract with a zero self-evident component. Third, deviations that lead to distributions of $(\widetilde{a}, \widehat{y})$ that cannot be caused by some other player can be deterred by a zero-sum contract. In the terminology of Rahman and Obara (2010), a deviating strategy profile $(\sigma_1, \ldots, \sigma_n)$ satisfying

$$\pi^{\sigma_1} = \cdots = \pi^{\sigma_n}$$

is *unattributable*, as the common distribution $\pi^{\sigma_1}$ could have been caused by any player. They show that if there is no unattributable deviating profile, then $\eta$ can be enforced by a contract with total transfer summing to zero. By comparison, Theorem 3A shows that $L^*(\eta) = 0$ if all unattributable deviations are detectable with respect to some $\omega \in P$ (i.e., changing the distribution of $(\widetilde{a}, \widehat{y})$ conditional on $\omega \in P$).

Let

$$Q(\eta) \equiv \{\sigma \in \Sigma | \pi^{\sigma_1} = \cdots = \pi^{\sigma_n} \in \mathrm{co}(\{\mu(\cdot|\omega) \mid \omega \in P\})\}$$

denote the set of unattributable deviations that are undetectable with respect to any $\omega \in P$. If $\sigma_i$ does not satisfy any of Conditions (i) to (iii) in Theorem 3A, then there must exist $\sigma_{-i}$ such that $(\sigma_i, \sigma_{-i})$ belongs to $Q(\eta)$. The next theorem characterizes $L^*(\eta)$ when $Q(\eta)$ is nonempty. To state the theorem, we need some extra notations. For any $\sigma_i$, we use

$$d(\sigma_i) \equiv \sum_{(\alpha_i, \rho_i)} \sigma_i(\alpha_i, \rho_i) \sum_{\tilde{a} \in A} \left(g_i\left(\tilde{a}_{-i}, \alpha_i(\tilde{a}_i)\right) - g_i(\tilde{a})\right) \eta\left(\tilde{a}\right)$$

to denote player $i$'s gain from the deviation $\sigma_i$, and

$$l(\sigma_i) \equiv \max_{\omega \in P} \frac{\pi^{\sigma_i}(\omega)}{\mu(\omega)}$$

to measure the difference between $\pi^{\sigma_i}$ and $\mu$.[15] Let

$$K(\eta) \equiv \{\sigma \in \Sigma | \pi^{\sigma_1} = \cdots = \pi^{\sigma_n} = \mu\}$$

denote the set of unattributable and undetectable strategy profiles.

**Theorem 3B.** For any enforceable $\eta$,

$$L^*(\eta) = \sup_{(\sigma_1, \ldots, \sigma_n) \in Q(\eta)/K(\eta)} \frac{\max\left(\sum_{i=1}^n d\left(\sigma_i\right), 0\right)}{l\left(\sigma_1\right) - 1}$$

if $Q(\eta)/K(\eta)$ is nonempty. Otherwise, $L^*(\eta) = 0$.

Theorem 3B implies that $L^*(\eta) > 0$ only if there is some $\sigma \in Q(\eta)/K(\eta)$ with $\sum_{i=1}^n d\left(\sigma_i\right) > 0$. Theorem 3B, thus, implies Theorem 3A. To illustrate Theorem 3B, consider some $\sigma \in Q(\eta)/K(\eta)$. Any $w$ that enforces $\eta$ must satisfy, for all player $i$, the incentive-compatibility constraint that

$$\sum_{(\tilde{a}, \hat{y}) \in A \times Y} \left(\mu\left(\tilde{a}, \hat{y}\right) - \pi^{\sigma_i}\left(\tilde{a}, \hat{y}\right)\right) w_i\left(\tilde{a}, \hat{y}\right) \geq d\left(\sigma_i\right). \tag{15}$$

[15] In the noisy Prisoner's Dilemma example, $l(\sigma_1) = (1-q)/(1-p)$.

Since $\sigma \in Q(\eta)$, for all $\omega \in P$,

$$E\left[w_i\left(\widetilde{a},\widehat{y}\right)|\mu,\omega\right] = E\left[w_i\left(\widetilde{a},\widehat{y}\right)|\pi^{\sigma_i},\omega\right]. \tag{16}$$

Substituting (16) into (15), and summing over $i$, we have

$$\sum_{\omega\in P}\left(\mu\left(\omega\right)-\pi^{\sigma_i}\left(\omega\right)\right)\sum_{i=1}^{n}E\left[w_i\left(\widetilde{a},\widehat{y}\right)|\mu,\omega\right] \geq \sum_{i=1}^{n}d\left(\sigma_i\right). \tag{17}$$

It then follows from the definition of $L$, (6), that

$$\begin{aligned}
L\left(\eta,w\right) &= \sum_{\omega\in P}\mu\left(\omega\right)\left(-\sum_{i=1}^{n}E\left[w_i\left(\widetilde{a},\widehat{y}\right)|\mu,\omega\right]+\max_{\omega\in P}\sum_{i=1}^{n}E\left[w_i\left(\widetilde{a},\widehat{y}\right)|\mu,\omega\right]\right)\\
&\geq \sum_{\omega\in P}\frac{\pi^{\sigma_i}\left(\omega\right)-\mu\left(\omega\right)}{l\left(\sigma_1\right)-1}\left(-\sum_{i=1}^{n}E\left[w_i\left(\widetilde{a},\widehat{y}\right)|\mu,\omega\right]+\max_{\omega\in P}\sum_{i=1}^{n}E\left[w_i\left(\widetilde{a},\widehat{y}\right)|\mu,\omega\right]\right)\\
&= \sum_{\omega\in P}\frac{\mu\left(\omega\right)-\pi^{\sigma_i}\left(\omega\right)}{l\left(\sigma_1\right)-1}\sum_{i=1}^{n}E\left[w_i\left(\widetilde{a},\widehat{y}\right)|\mu,\omega\right]\\
&\geq \frac{\sum_{i=1}^{n}d\left(\sigma_i\right)}{l\left(\sigma_1\right)-1}.
\end{aligned}$$

Intuitively, since $\sigma$ is unattributable, every player must be punished, and the total punishment must be greater than $\sum_{i=1}^{n}d\left(\sigma_i\right)$, the total deviating gain. The resulting efficiency loss is equal to the total punishment multiplied by a factor that measures the difference between $\pi^{\sigma_i}$ and $\mu$. The smaller the difference, the harder it is to distinguish between the two distributions, and the higher the efficiency loss.

Since the argument applies to every $w$ that enforces $\eta$ and every $\sigma \in Q(\eta)/K(\eta)$,

$$L^*\left(\eta\right) \geq \sup_{\left(\sigma_1,\ldots,\sigma_n\right)\in Q(\eta)/K(\eta)}\frac{\sum_{i=1}^{n}d\left(\sigma_i\right)}{l\left(\sigma_1\right)-1}.$$

In Appendix F, we show that the converse also holds.

# 7 Correlated Strategies

A natural way to improve the long-run efficiency is to change the signal structure, for example, by introducing private signals or removing public ones. More interestingly, given a fixed information structure, the players may also change the information structure endogenously by adopting a correlated strategy profile.[16] Since a small change in a correlated strategy can alter the support of the distribution of action-signal profiles substantially, it can have a large impact on the long-run efficiency loss as $\delta$ goes to one and $T$ goes to infinity. The idea is first raised by Rahman (2014). He proves a folk theorem under the condition that the signal distribution satisfies conditional identifiability.

To illustrate the idea, let us return to the noisy Prisoners' Dilemma in Section 3. As we showed in the last section, $L^*(C,C) = (1-p)d/(p-q)$. Consider the correlated strategy profile $\overline{\eta}$ where

$$\overline{\eta}(C,C) = 1-\varepsilon; \overline{\eta}(C,D) = \overline{\eta}(D,C) = 0.5\varepsilon.$$

When $\varepsilon$ is small, $\overline{\eta}$ is close to the pure strategy profile $(C,C)$. Yet the support of the distribution of the action-signal profiles under $\overline{\eta}$ is very different from the support under $(C,C)$. Now each $P_i$ consists of four elements. In particular,

$$\begin{aligned} P_1 &: \{CCH, CDH\}, \{CCL, CDL\}, \{DCH\}, \{DCL\} \\ P_2 &: \{CCH, DCH\}, \{CCL, DCL\}, \{CDH\}, \{CDL\}, \end{aligned}$$

and the meet of $P_1$ and $P_2$ is

$$P : \{CCH, CDH, DCH\}, \{CCL, CDL, DCL\}.$$

Note that each member of the meet contains multiple elements. As a player is instructed to choose $C$, he is not sure whether the other player is choosing $C$ or $D$. Let $p(H|DD) = r$. Conditional identifiability requires that

$$\frac{p}{q} \neq \frac{q}{r} \quad \text{and} \quad \frac{1-p}{1-q} \neq \frac{1-q}{1-r}.$$

The condition says that how player $i$'s action affects the relative likelihood of $H$ and $L$ depends on whether player $j$ is choosing $C$ or $D$. Since player $i$ does not observe the action of player $j$, $\overline{\eta}$ can be "secretly" enforced so that the players do not learn about their own punishments from the public signal. Hence, by choosing a small $\varepsilon$, the players

[16]This involves changing the correlating device correspondingly.

can obtain close to the efficient payoff $(1,1)$, when the players are sufficiently patient and $T$ is sufficiently large.

Using Theorems 2 and 3A, we can show that $\overline{\eta}$ can be enforced almost efficiently without assuming conditional identifiability. It is straightforward to see that $\overline{\eta}$ is enforceable. Let $\alpha_i^{xy}$ denote the strategy of choosing $x$ when $C$ is recommended and $y$ when $D$ is recommended. Each player has four pure action strategies: $\alpha_i^{CD}$, $\alpha_i^{DD}$, $\alpha_i^{CC}$, and $\alpha_i^{DC}$. In Table 3, each row gives the probabilities of outcomes with an $H$ signal under a different pure strategy of player 1 (assuming that player 2 plays $\alpha_2^{CD}$).

| | $CCH$ | $DCH$ | $CDH$ |
|---|---|---|---|
| $\alpha_1^{CD}$ | $(1-\varepsilon)p$ | $0.5\varepsilon q$ | $0.5\varepsilon q$ |
| $\alpha_1^{DD}$ | $(1-\varepsilon)q$ | $0.5\varepsilon q$ | $0.5\varepsilon r$ |
| $\alpha_1^{CC}$ | $(1-\varepsilon)p$ | $0.5\varepsilon p$ | $0.5\varepsilon q$ |
| $\alpha_1^{DC}$ | $(1-\varepsilon)q$ | $0.5\varepsilon p$ | $0.5\varepsilon r$ |

Table 3: The probability for each outcome with an $H$ signal.

Notice that the ratio of the relative probability of $CCH$ over $DCH$ is strictly higher when player 1 follows the recommendation and plays $\alpha_1^{CD}$. Hence, every deviation is detectable with respect to the self-evident event $\{CCH, CDH, DCH\}$. By Theorem 3A, $L^*(\overline{\eta}) = 0$. Intuitively, the recommendation $DC$ serves as a "benchmark" for player 1. Given that player 2 is choosing $C$, player 1 choosing $D$ minimizes the probability of $H$. Hence, if player 1 deviates to $D$ when told to choose $C$, he must lower the relative probability of $CCH$ over $DCH$. Finally, since any unilateral deviation from $\overline{\eta}$ is detectable, $\overline{\eta}$ satisfies the no-free-information condition. Hence, by Theorem 2, $\overline{\eta}$ can be enforced almost efficiently in the long run.

The following theorem generalizes the above example. It says that any strictly enforceable outcome can be virtually enforced with almost no long-run efficiency loss.

**Theorem 4.** *For any strictly enforceable $\eta$ and any $\varepsilon > 0$, there exists an enforceable correlated action profile $\overline{\eta}$ that satisfies the no-free-information condition and with $\max_{\widetilde{a}\in A} |\eta(\widetilde{a}) - \overline{\eta}(\widetilde{a})| \leq \varepsilon$ and $L^*(\overline{\eta}) = 0$.*

We prove Theorem 4 in two steps. First, given a strictly enforceable $\eta$, we construct a correlated outcome $\overline{\eta}$ close to $\eta$ by adding "benchmarks" so that any unilateral deviation is detectable with respect to some element of the meet induced by $\overline{\eta}$. We then complete the proof by showing that $\overline{\eta}$ is enforceable and satisfies the no-free-information condition. Note that Theorem 4 does not hold if $\eta$ merely satisfies the no-free-information condition but is not strictly enforceable. The strict enforceability of $\eta$, together with the fact that $\overline{\eta}$ is close to $\eta$, ensures that under $\overline{\eta}$ no player can deviate undetectably without strictly reducing his stage-game payoff when recommended to choose an action in the support of $\eta$.

# 8 Infinitely-Repeated Game with Mediated Communication

Our results can be used to obtain the maximum equilibrium payoff in repeated games with mediated communication and side-payments. Formally, consider a repeated game where the stage game unfolds as follows.

(i) The correlating device draws $\widetilde{a}$ from $A$ according to some distribution on $A$ and then privately informs each player $i$ of $\widetilde{a}_i$.

(ii) Each player $i$ simultaneously chooses a private action $a_i$ from $A_i$.

(iii) Nature draws $y$ from $Y$ according to the distribution $p(\cdot|a) \in \Delta(Y)$. Each player $i$ privately observes $y_i$.

(iv) Each player $i$ publicly reports $m_i \in M_i$ and the correlating device publicly reports $m_0 \in M_0$.

(v) Each player $i$ then simultaneously makes a publicly observable side-payment $\tau_{ij}$ to each player $j$.

(vi) The players observe the outcome of a public randomization device that is uniformly distributed between 0 and 1.

Suppose the signal distribution $p$ has invariant support, and the message spaces are sufficiently rich that the players and the correlating device can withhold the private signals and recommendations for any number of periods and then reveal them all together. Players discount future payoffs by $\delta$ as in the $T$-period game.

Let

$$v' \equiv \sup_{\eta\colon \text{strictly enforceable}} \sum_{i=1}^{n} \sum_{\widetilde{a}\in A} g_i(\widetilde{a})\eta(\widetilde{a}).$$

Following Chan and Zhang (2016), we can prove the following:

*For any $\varepsilon > 0$, there exists $\bar{\delta} < 1$ such that for any $\delta > \bar{\delta}$, there is a perfect $T$-public equilibrium with total average payoff greater than $v' - \varepsilon$.*

The lower bound $v'$ follows from applying Theorem 4 to construct a $T$-period linking mechanism and then embedding this mechanism into the infinitely-repeated game as in Chan and Zhang (2016).

# 9 Conclusion

Players in a long-run relationship can reduce incentive costs by linking incentives across periods, but the value of linking is limited by the information the players obtain during the course of the relationship. We show that the long-run per-period efficiency loss in enforcing an action profile is bounded from below by the incentive cost that becomes self-evident at the end of each period, and the bound is tight when players cannot obtain free information undetectably. The results extend the insights of Abreu, Milgrom, and Pearce (1991) to general stage games where players may observe both public and private signals and use a correlating device to coordinate their actions.

In this paper we have focused on repeated games. But the notion of self-evident events is inherently a dynamic concept that describes the knowledge structure of the players at each point in time. In future works, we plan to apply the same concept to understand dynamic games where actions have long-term consequences or where information about actions is revealed gradually over time.

# A Proof of Theorem 1

We proceed by induction on $T$. By definition, $W^*(\eta) \geq L^*(\eta)$. Assume that $W^*(\eta, T-1, \delta) \geq L^*(\eta)$ and consider the $T$-period case.

For each variable $x = \widetilde{a}$, $\widehat{y}$, let $x^{2,T}$ denote the value of $x$ from period 2 through $T$. Fix $w^T \in \mathscr{W}(\eta, T, \delta)$. For each $i$ and each $(\widetilde{a}(1), \widehat{y}(1)) \in A \times Y$, we let

$$w_i(\widetilde{a}(1), \widehat{y}(1)) \equiv \sum_{(\widetilde{a}^{2,T}, \widehat{y}^{2,T})} w_i^T(\widetilde{a}(1), \widetilde{a}^{2,T}, \widehat{y}(1), \widehat{y}^{2,T}) \prod_{t=2}^{T} \mu(\widetilde{a}(t), \widehat{y}(t))$$

denote the expected value of $w_i^T$ conditional on $(\widetilde{a}(1), \widehat{y}(1))$, assuming that all players follow the equilibrium strategy. For each $i$, each $\left(\widetilde{a}^{2,T}, \widehat{y}^{2,T}\right)$ and each $\omega \in P$, we let

$$w_i^{T-1,\omega}\left(\widetilde{a}^{2,T}, \widehat{y}^{2,T}\right) \equiv \sum_{(\widetilde{a}(1), \widehat{y}(1)) \in \omega} w_i\left(\widetilde{a}^T, \widehat{y}^T\right) \mu\left(\widetilde{a}(1), \widehat{y}(1) | \, \omega\right)$$

denote the expected value of $w_i^T$ conditional on $(\widetilde{a}(1), \widehat{y}(1))$ being in the set $\omega$ and $\left(\widetilde{a}^{2,T}, \widehat{y}^{2,T}\right)$, assuming that all players follow the equilibrium strategy.

Since $w^T \in \mathscr{W}(\eta, T, \delta)$, we have $w \in \mathscr{W}(\eta)$ and $\delta^{-1} w^{T-1,\omega} \in \mathscr{W}(\eta, T-1, \delta)$. Let

$$\omega^* \in \arg\max_{\omega \in P} \sum_{i=1}^{n} E\left[w_i(\widetilde{a}(1), \widehat{y}(1)) \,|\, \sigma^*, \omega\right].$$

It follows that

$$
\begin{aligned}
& -\sum_{i=1}^{n} E[w_i^T(\widetilde{a}^T,\widehat{y}^T)|\sigma^{T*}] \\
= & -\sum_{i,(\widetilde{a}^T,\widehat{y}^T)} w_i^T(\widetilde{a}^T,\widehat{y}^T)\prod_{t=1}^{T}\mu(\widetilde{a}(t),\widehat{y}(t)) \\
= & -\sum_{i,(\widetilde{a}(1),\widehat{y}(1))}\left(\sum_{(\widetilde{a}^{2,T},\widehat{y}^{2,T})} w_i^T(\widetilde{a}^T,\widehat{y}^T)\prod_{t=2}^{T}\mu(\widetilde{a}(t),\widehat{y}(t))\right)\mu(\widetilde{a}(1),\widehat{y}(1)) \\
= & -\sum_{i,(\widetilde{a}(1),\widehat{y}(1))} w_i(\widetilde{a}(1),\widehat{y}(1))\mu(\widetilde{a}(1),\widehat{y}(1)) \\
\geq & \ L^*(\eta)-\sum_{i,(\widetilde{a}(1),\widehat{y}(1))\in\omega^*} w_i(\widetilde{a}(1),\widehat{y}(1))\mu(\widetilde{a}(1),\widehat{y}(1)|\omega^*) \\
= & \ L^*(\eta)-\sum_{i,(\widetilde{a}(1),\widehat{y}(1))\in\omega^*}\left(\sum_{(\widetilde{a}^{2,T},\widehat{y}^{2,T})} w_i^T(\widetilde{a}^T,\widehat{y}^T)\prod_{t=2}^{T}\mu(\widetilde{a}(t),\widehat{y}(t))\right)\mu(\widetilde{a}(1),\widehat{y}(1)|\omega^*) \\
= & \ L^*(\eta)-\sum_{i,(\widetilde{a}^{2,T},\widehat{y}^{2,T})}\left(\sum_{(\widetilde{a}(1),\widehat{y}(1))\in\omega^*} w_i^T(\widetilde{a}^T,\widehat{y}^T)\mu(\widetilde{a}(1),\widehat{y}(1)|\omega^*)\right)\prod_{t=2}^{T}\mu(\widetilde{a}(t),\widehat{y}(t)) \\
= & \ L^*(\eta)-\sum_{i,(\widetilde{a}^{2,T},\widehat{y}^{2,T})} w_i^{T-1,\omega^*}(\widetilde{a}^{2,T},\widehat{y}^{2,T})\prod_{t=2}^{T}\mu(\widetilde{a}(t),\widehat{y}(t)) \\
\geq & \ L^*(\eta)+\delta\frac{1-\delta^{T-1}}{1-\delta}L^*(\eta) \\
= & \ \frac{1-\delta^T}{1-\delta}L^*(\eta).
\end{aligned}
$$

# B Proof of Lemma 2

The first part of the lemma is obvious. We prove the second part. By Lemma 1, pure and undetectable strategies $(\alpha_i,\rho_i)$ are unprofitable. Hence, it suffices to show that there exists a contract $w$ such that for any player $i$ and any pure strategy $(\alpha_i,\rho_i)$,

$$v_i(\sigma^*;w_i)\geq v_i(\sigma^*_{-i},\alpha_i,\rho_i;w_i),$$

with the inequality strict if $(\alpha_i, \rho_i)$ is detectable. That is, for any player $i$, there exists $w_i$ such that

$$\sum_{(\widetilde{a},\widehat{y})\in A\times Y} [\mu(\widetilde{a},\widehat{y}) - \pi^{\alpha_i,\rho_i}(\widetilde{a},\widehat{y})]\, w_i(\widetilde{a},\widehat{y}) \geq \sum_{\widetilde{a}\in A} (g_i(\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) - g_i(\widetilde{a}))\eta(\widetilde{a}) \qquad (18)$$

for all $(\alpha_i, \rho_i)$, and the inequality is strict if $\pi^{\alpha_i,\rho_i} \neq \mu$. By the theory of alternatives (see, e.g., Proposition 5.6.2 of Bertsekas, 2009), (18) does not have a solution $w_i$ if and only if there exists $\lambda_i(\alpha_i, \rho_i) \geq 0$ for each $(\alpha_i, \rho_i)$ such that

$$\sum_{(\alpha_i,\rho_i)} \lambda_i(\alpha_i,\rho_i)\, [\mu(\widetilde{a},\widehat{y}) - \pi^{\alpha_i,\rho_i}(\widetilde{a},\widehat{y})] = 0 \text{ for each } (\widetilde{a},\widehat{y}),$$

and either one of the following two cases holds:

(i) We have $\sum_{(\alpha_i,\rho_i)} \lambda_i(\alpha_i,\rho_i) \sum_{\widetilde{a}\in A} (g_i(\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) - g_i(\widetilde{a}))\eta(\widetilde{a}) > 0$.

(ii) We have $\sum_{(\alpha_i,\rho_i)} \lambda_i(\alpha_i,\rho_i) \sum_{\widetilde{a}\in A} (g_i(\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) - g_i(\widetilde{a}))\eta(\widetilde{a}) \geq 0$ and $\lambda_i(\alpha_i,\rho_i) > 0$ for some $(\alpha_i,\rho_i)$ such that $\pi^{\alpha_i,\rho_i} \neq \mu$.

In either case, $\lambda_i(\alpha_i', \rho_i') > 0$ for some $(\alpha_i', \rho_i')$. By dividing each $\lambda_i(\alpha_i, \rho_i)$ by $\sum_{(\alpha_i',\rho_i')} \lambda_i(\alpha_i', \rho_i')$ if necessary, we may assume that $\sum_{(\alpha_i',\rho_i')} \lambda_i(\alpha_i', \rho_i') = 1$. That is, $\lambda_i$ represents a mixed strategy for player $i$.

Since $\eta$ is enforceable, by Lemma 1, Case (i) is impossible. Case (ii) violates the no-free-information condition. Hence (18) must have a solution.

# C Proof of Lemma 3

W.l.o.g., assume that $i = n$ and $P$ is singleton. Rewrite the conditions of the lemma as

$$\sum_{j\neq n} \sum_{(\widetilde{a}_{-n},\widehat{y}_{-n})} z_{n,j}(\widetilde{a},\widehat{y})\mu(\widetilde{a},\widehat{y}) = (n-1) \sum_{(\widetilde{a}_{-n},\widehat{y}_{-n})} w^*_{n,a}(\widetilde{a},\widehat{y})\mu(\widetilde{a},\widehat{y}) \quad \text{for all } (\widetilde{a}_n,\widehat{y}_n) \in A_n \times Y_n \qquad (19)$$

$$\sum_{(\widetilde{a}_{-j},\widehat{y}_{-j})} z_{n,j}(\widetilde{a},\widehat{y})\mu(\widetilde{a},\widehat{y}) = \mu_j(\widetilde{a}_j,\widehat{y}_j) \sum_{(\widetilde{a}',\widehat{y}')} w^*_{n,a}(\widetilde{a}',\widehat{y}')\mu(\widetilde{a}',\widehat{y}') \quad \text{for all } (\widetilde{a}_j,\widehat{y}_j) \text{ and } j \neq n, \qquad (20)$$

where $\mu_j$ denotes the marginal distribution of $\mu$ on $A_j \times Y_j$. To shorten notation, let $q_j$ denote a pair of recommendation and signal-report $(\widetilde{a}_j, \widehat{y}_j)$ and let $q = (q_1, \ldots, q_n)$. Let

$$I_{q_j}(q) = \begin{cases} 1, & \text{if the } j\text{th component of } q \text{ is } q_j \\ 0, & \text{otherwise.} \end{cases}$$

Then we can rewrite (19) and (20) as

$$\begin{aligned} \sum_{j \neq n} \sum_{q} z_{n,j}(q)\mu(q) I_{q_n}(q) &= f_n(q_n) \quad \text{for all } q_n \in A_n \times Y_n && (21) \\ \sum_{q} z_{n,j}(q)\mu(q) I_{q_j}(q) &= f_j(q_j) \quad \text{for all } q_j \in A_j \times Y_j \text{ and } j \neq n, && (22) \end{aligned}$$

where

$$\begin{aligned} f_n(q_n) &= (n-1) \sum_{q} w^*_{n,a}(q)\mu(q) I_{q_n}(q) \\ f_j(q_j) &= \mu_j(q_j) \sum_{q'} w^*_{n,a}(q')\mu(q') \quad \text{for each } j \neq n. \end{aligned}$$

For each $j \in N$, fix an enumeration $(q_j^1, q_j^2, \ldots, q_j^{m_j})$ of $A_j \times Y_j$. Let $m = \prod_{j=1}^{n} m_j$. Fix an enumeration $(q^1, q^2, \ldots, q^m)$ of $A \times Y$. For each $j$, let $B_j$ be the $m_j \times m$ matrix where the $(k,l)$ entry is $\mu(q^l) I_{q_j^k}(q^l)$. Rewrite (22) and (21) as

$$\begin{pmatrix} B_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & B_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & B_{n-1} \\ B_n & B_n & \ldots & B_n \end{pmatrix} \begin{pmatrix} z_{n,1}(q^1) \\ \vdots \\ z_{n,1}(q^m) \\ z_{n,2}(q^1) \\ \vdots \\ z_{n,2}(q^m) \\ \vdots \\ z_{n,n-1}(q^1) \\ \vdots \\ z_{n,n-1}(q^m) \end{pmatrix} = \begin{pmatrix} f_1(q_1^1) \\ \vdots \\ f_1(q_1^{m_1}) \\ f_2(q_2^1) \\ \vdots \\ f_2(q_2^{m_2}) \\ \vdots \\ f_{n-1}(q_{n-1}^1) \\ \vdots \\ f_{n-1}(q_{n-1}^{m_{n-1}}) \\ f_n(q_n^1) \\ \vdots \\ f_n(q_n^{m_n}) \end{pmatrix}. \tag{23}$$

By the Fredholm alternative (Theorem 2.5, Gale, 1960), either (23) has a solution $(z_{n,1},\ldots,z_{n,n-1})$, or there exists $\lambda_{j,k}$ for each $j \in N$ and $k \leq m_j$ such that

$$\left(\lambda_{j,1},\ldots,\lambda_{j,m_j}\right) B_j + (\lambda_{n,1},\ldots,\lambda_{n,m_n}) B_n = (0,\ldots,0) \quad \text{for each } j \neq n \tag{24}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{m_j} \lambda_{j,k} f_j(q_j^k) \neq 0. \tag{25}$$

We show that (24) and (25) are incompatible. Hence, (23) has a solution.

Rewrite the $l$th column of (24) as

$$\sum_{k=1}^{m_j} \lambda_{j,k}\mu(q^l) I_{q_j^k}(q^l) + \sum_{k=1}^{m_n} \lambda_{n,k}\mu(q^l) I_{q_n^k}(q^l) = 0 \quad \text{for each } j \neq n. \tag{26}$$

Let $\zeta_j(l) \leq m_j$ denote the index such that the $j$th component of $q^l$ is $q_j^{\zeta_j(l)}$. Then (26) is equivalent to

$$\mu(q^l)\left(\lambda_{j,\zeta_j(l)} + \lambda_{n,\zeta_n(l)}\right) = 0 \quad \text{for each } j \neq n. \tag{27}$$

Suppose that (27) holds. Fix any $l'$ and $l''$ such that $\mu(q^{l'}) > 0$ and $\mu(q^{l''}) > 0$. Since the meet $P$ is singleton, $q^{l''}$ is reachable from $q^{l'}$, i.e., there exists a sequence action-signal profiles $q^{l'} = q^{l_0}, q^{l_1}, \ldots, q^{l_d} = q^{l''}$ such that $\mu(q^{l_s}) > 0$ for each $s \leq d$ and any two consecutive profiles $q^{l_{s-1}}$ and $q^{l_s}$ have the same $i_s$th component for some player $i_s \in N$ (see, e.g., Aumann, 1976; Geanakoplos, 1994). Hence, $\zeta_{i_s}(l_{s-1}) = \zeta_{i_s}(l_s)$. If $i_s = n$, then $\lambda_{n,\zeta_n(l_{s-1})} = \lambda_{n,\zeta_n(l_s)}$. If $i_s \neq n$, then (27) implies that

$$\lambda_{n,\zeta_n(l_{s-1})} = -\lambda_{i_s,\zeta_{i_s}(l_{s-1})} = -\lambda_{i_s,\zeta_{i_s}(l_s)} = \lambda_{n,\zeta_n(l_s)}.$$

It follows that $\lambda_{n,\zeta_n(l_0)} = \lambda_{n,\zeta_n(l_d)}$. Since $l'$ and $l''$ are chosen arbitrarily, this implies that there exists a constant $\lambda$ such that

$$\lambda_{n,\zeta_n(l)} = \lambda \qquad \text{for all } l \leq m \text{ such that } \mu(q^l) > 0. \tag{28}$$

Then (28) and (27) imply that

$$\lambda_{j,\zeta_j(l)} = -\lambda \qquad \text{for all } l \leq m \text{ such that } \mu(q^l) > 0 \text{ and all } j \neq n. \tag{29}$$

Using (28) and (29), it is straightforward to verify that

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=1}^{m_j}\lambda_{j,k}f_j(q_j^k)\\
=&-\lambda\sum_{j=1}^{n-1}\sum_{k=1}^{m_j}\mu_j(q_j^k)\sum_{l=1}^{m}w^*_{n,a}(q^l)\mu(q^l)+\lambda(n-1)\sum_{k=1}^{m_n}\sum_{l=1}^{m}w^*_{n,a}(q^l)\mu(q^l)I_{q_n^k}(q^l)\\
=&-\lambda\sum_{j=1}^{n-1}\sum_{l=1}^{m}w^*_{n,a}(q^l)\mu(q^l)+\lambda(n-1)\sum_{l=1}^{m}w^*_{n,a}(q^l)\mu(q^l)\\
=&0,
\end{aligned}$$

which contradicts (25).

# D Proof of Lemma 4

The term $R_i^+(1-I_{i,i})$ is strictly positive only when

$$w_{i,a}^{T*}(\widetilde{a}^T,\widehat{y}^T)>E[w_{i,a}^{T*}(\widetilde{a}^{T\prime},\widehat{y}^{T\prime})|\sigma^{T*}]+\kappa_T \text{ and } z_{i,i}^T(\widetilde{a}^T,\widehat{y}^T)\le E[w_{i,a}^{T*}(\widetilde{a}^{T\prime},\widehat{y}^{T\prime})|\sigma^{T*}]+\frac{1}{2}\kappa_T,$$

which implies

$$w_{i,a}^{T*}(\widetilde{a}^T,\widehat{y}^T)-z_{i,i}^T(\widetilde{a}^T,\widehat{y}^T)>\frac{1}{2}\kappa_T. \tag{30}$$

The term $\sum_{j\neq i}R_j^+I_{j,i}$ is strictly positive only when

$$z_{j,i}^T(\widetilde{a}^T,\widehat{y}^T)>E[w_{j,a}^{T*}(\widetilde{a}^{T\prime},\widehat{y}^{T\prime})|\sigma^{T*}]+\frac{1}{2}\kappa_T \quad \text{for some player } j\neq i. \tag{31}$$

By Lemma 3, the means of the left-hand sides of (30) and (31) conditional on any $h_i\in H_i^{T*}$ are 0 and $E[w_{j,a}^{T*}(\widetilde{a}^{T\prime},\widehat{y}^{T\prime})|\sigma^{T*}]$, respectively. Hence, the events in (30) and (31) are deviations from the means by $\frac{1}{2}\kappa_T$.

We apply the following inequality of Hoeffding (1963) to provide upper bounds for the probabilities of these events. Suppose that $\xi(1)$, $\xi(2)$, …, $\xi(T)$ are independent random variables such that $|\xi(t)|\le\nu$ for each $t\le T$. Then, for any $\kappa>0$, Hoeffding's inequality asserts that

$$\Pr\left(\sum_{t=1}^{T}\xi(t)\ge E\left[\sum_{t=1}^{T}\xi(t)\right]+\kappa\right)\le\exp\left(-\frac{\kappa^2}{2\nu^2T}\right).$$

Let $c$ be a bound for the following stage-game variables

$$c = \max\{|w^*_{i,a}(\widetilde{a},\widehat{y})|, |z_{i,j}(\widetilde{a},\widehat{y})|, |w^*_{i,a}(\widetilde{a},\widehat{y}) - z_{i,i}(\widetilde{a},\widehat{y})| : i, j \in N, \widetilde{a} \in A, \widehat{y} \in Y\}.$$

Substituting $\delta^{t-1}(w^*_{i,a}(\widetilde{a}(t),\widehat{y}(t)) - z_{i,i}(\widetilde{a}(t),\widehat{y}(t)))$ for $\xi(t)$, $c$ for $\nu$, and $\frac{1}{2}\kappa_T$ for $\kappa$ in Hoeffding's inequality, we have

$$\Pr\left( w^{T*}_{i,a}(\widetilde{a}^T,\widehat{y}^T) - z^T_{i,i}(\widetilde{a}^T,\widehat{y}^T) > \frac{1}{2}\kappa_T \,\middle|\, \sigma^{T*}, h_i \right) \leq \exp\left(-\frac{\kappa_T^2}{8c^2T}\right).$$

Since $R_i^+(\widetilde{a}^T,\widehat{y}^T) \leq cT$,

$$E[R_i^+(\widetilde{a}^T,\widehat{y}^T)\left(1 - I_{i,i}(\widetilde{a}^T,\widehat{y}^T)\right)|\sigma^{T*}, h_i] \leq cT \exp\left(-\frac{\kappa_T^2}{8c^2T}\right). \tag{32}$$

Substituting $\delta^{t-1}z_{j,i}(\widetilde{a}(t),\widehat{y}(t))$ for $\xi(t)$, $c$ for $\nu$, and $\frac{1}{2}\kappa_T$ for $\kappa$ in Hoeffding's inequality, we have

$$\Pr\left( z^T_{j,i}\left(\widetilde{a}^T,\widehat{y}^T\right) > E[w^{T*}_{j,a}(\widetilde{a}^{T\prime},\widehat{y}^{T\prime})|\sigma^{T*}] + \frac{1}{2}\kappa_T \,\middle|\, \sigma^{T*}, h_i \right) \leq \exp\left(-\frac{\kappa_T^2}{8c^2T}\right).$$

Hence,

$$\sum_{j\neq i} E[R_j^+(\widetilde{a}^T,\widehat{y}^T) I_{j,i}(\widetilde{a}^T,\widehat{y}^T)|\sigma^{T*}, h_i] \leq (n-1)cT \exp\left(-\frac{\kappa_T^2}{8c^2T}\right). \tag{33}$$

Combining (32) and (33), we have

$$E\left[ R_i^+(\widetilde{a}^T,\widehat{y}^T)(1 - I_{i,i}(\widetilde{a}^T,\widehat{y}^T)) + \sum_{j\neq i} R_j^+(\widetilde{a}^T,\widehat{y}^T) I_{j,i}(\widetilde{a}^T,\widehat{y}^T)|\sigma^{T*}, h_i \right] \leq ncT \exp\left(-\frac{\kappa_T^2}{8c^2T}\right).$$

Since $\kappa_T = T^{2/3}$, $ncT\exp(-\frac{\kappa_T^2}{8c^2T}) = ncT\exp(-\frac{T^{1/3}}{8c^2})$, which tends to 0 as $T$ tends to infinity.

# E Proof of Theorem 2

Since $w^*$ enforces $\eta$ almost strictly, and the set of pure strategies $(\alpha_i, \rho_i)$ is finite, there exists $\Delta_0 > 0$ such that, for all detectable strategy $(\alpha_i, \rho_i)$,

$$v_i(\sigma^*; w_i^*) - v_i(\sigma^*_{-i}, \alpha_i, \rho_i; w_i^*) > \Delta_0. \tag{34}$$

Since $\eta$ satisfies the no-free-information condition, there exists $\Delta_1 > 0$ such that if $a_i$ is strictly more informative than $\widetilde{a}_i$ at $\widetilde{a}_i$, then

$$\sum_{\widetilde{a}_{-i}}(g_i(\widetilde{a}) - g_i(\widetilde{a}_{-i}, a_i))\eta(\widetilde{a}) > \Delta_1. \tag{35}$$

Let $\Delta_2 = \min\{\Delta_0, \Delta_1\}$ and let $\varepsilon_T = ncT \exp\left(-\frac{T^{1/3}}{8c^2}\right)$.

**Claim 1.** *Suppose that $\delta^{T-1}\Delta_2 \geq \varepsilon_T$. Then $w^{T**}$ enforces $\eta$ for $T$ periods.*

Choose $T_0$ large enough such that for all $T \geq T_0$ and $\delta \geq 1 - T^{-2}$,

$$\begin{aligned} \varepsilon_T &\leq \delta^{T-1}\Delta_2 \\ n\frac{1-\delta}{1-\delta^T}\left(\varepsilon_T + T^{2/3}\right) &\leq \frac{\varepsilon}{2}. \end{aligned}$$

Then, by Claim 1, $w^{T**}$ enforces $\eta$ for $T$ periods. By (14), the per-period efficiency loss is

$$\begin{aligned} W\left(\eta, T, \delta, w^{T**}\right) =& \frac{1-\delta}{1-\delta^T}\sum_{i=1}^{n} -E\left[w_i^{T**}|\sigma^{T*}\right] \\ \leq& L(\eta, w^*) + n\frac{1-\delta}{1-\delta^T}\left(\varepsilon_T + T^{2/3}\right) \leq L^*(\eta) + \varepsilon. \end{aligned}$$

It remains to prove Claim 1.

We say that a $T$-period action strategy $\alpha_i^T$ induces a stage-game action strategy $\alpha_i$ at a history $h_i = (\widetilde{a}_i^{t-1}, a_i^{t-1}, y_i^{t-1}) \in H_i^{T*}$ if $\alpha_i(\widetilde{a}_i(t)) = \alpha_i^T(h_i, \widetilde{a}_i(t))$ for all $\widetilde{a}_i(t)$.

We say that $\alpha_i$ is equivalent to $\alpha_i^*$ if there exists a reporting strategy $\rho_i$ such that $(\alpha_i, \rho_i)$ is undetectable and, for any $\widetilde{a}$ in the support of $\eta$ and any $y_i'$ and $y_i''$ such that $p(y_i'|\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) > 0$ and $p(y_i''|\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) > 0$, we have $\rho_i(\widetilde{a}_i, \alpha_i(\widetilde{a}_i), y_i') \neq \rho_i(\widetilde{a}_i, \alpha_i(\widetilde{a}_i), y_i'')$. This implies that for any $y_i$, there is a unique $y_i'$ such that

$$p\left(y_{-i}, y_i'|\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)\right) = p\left(y_{-i}, y_i|\widetilde{a}\right) \text{ for all } y_{-i}.$$

We proceed in two steps to show that $\sigma_i^{T*}$ is a best response against $\sigma_{-i}^{T*}$ under $w_i^{T**}$.

In Step 1, we show that if $\alpha_i^T$ induces a strategy $\alpha_i$ equivalent to but not equal to $\alpha_i^*$ at a history $h_i = (\widetilde{a}_i^{t-1}, a_i^{t-1}, y_i^{t-1}) \in H_i^{T*}$, then for any $T$-period reporting strategy $\rho_i^T$, the pair $(\alpha_i^T, \rho_i^T)$ is weakly dominated by some other strategy.

In Step 2, we show that any strategy $(\alpha_i^T, \rho_i^T)$ satisfying one of the following three conditions is strictly dominated by $\sigma_i^{T*}$.

(i) There exists a history $h_i = \left(\widetilde{a}_i^{t-1}, a_i^{t-1}, y_i^{t-1}\right) \in H_i^{T*}$ at which $\alpha_i^T$ induces a strategy $\alpha_i$ not equivalent to $\alpha_i^*$ and $(\alpha_i, \rho_i)$ is undetectable for some $\rho_i$.

(ii) There exists a history $h_i = \left(\widetilde{a}_i^{t-1}, a_i^{t-1}, y_i^{t-1}\right) \in H_i^{T*}$ at which $\alpha_i^T$ induces a strategy $\alpha_i$ such that $(\alpha_i, \rho_i)$ is detectable for any $\rho_i$.

(iii) We have $\alpha_i^T = \alpha_i^{T*}$ and $\rho_i^T(h_i) \neq \rho_i^{T*}(h_i)$ for some $h_i = \left(\widetilde{a}_i^T, a_i^T, y_i^T\right) \in H_i^{T*}$.

By Step 1, it suffices to consider strategies $(\alpha_i^T, \rho_i^T)$ where $\alpha_i^T$ never induces a strategy equivalent to but not equal to $\alpha_i^*$ at a history in $H_i^{T*}$. But then $(\alpha_i^T, \rho_i^T)$ must satisfy one of the conditions in Step 2 if $(\alpha_i^T, \rho_i^T) \neq \sigma_i^{T*}$, hence it is strictly dominated by $\sigma_i^{T*}$. This proves Claim 1.

## E.1 Step 1

Define a strategy $(\overline{\alpha}_i^T, \overline{\rho}_i^T)$ that dominates $(\alpha_i^T, \rho_i^T)$ as follows. We modify $(\alpha_i^T, \rho_i^T)$ so that after $h_i$ player $i$ follows $\alpha_i^*$ in period $t$ instead of $\alpha_i$, and then conditional on a recommendation $\widetilde{a}_i(t)$ and a signal $y_i(t)$, he follows the continuation strategy at the history $(h_i, \widetilde{a}_i(t), \alpha_i(\widetilde{a}_i(t)), y_i')$ instead of that at $(h_i, \widetilde{a}_i(t), \widetilde{a}_i(t), y_i(t))$, where $y_i'$ is the signal such that $y_i(t) = \rho_i(\widetilde{a}_i(t), \alpha_i(\widetilde{a}_i(t)), y_i')$ and $\rho_i$ is the reporting strategy such that $(\alpha_i, \rho_i)$ is undetectable. Since $\alpha_i$ is equivalent to $\alpha_i^*$, there is a unique $y_i'$ satisfying this condition. Let $(\overline{\alpha}_i^T, \overline{\rho}_i^T)$ be identical with $(\alpha_i^T, \rho_i^T)$ following other histories. Clearly, $(\overline{\alpha}_i^T, \overline{\rho}_i^T)$ and $(\alpha_i^T, \rho_i^T)$ produce the same distribution of $(\widetilde{a}^T, \widehat{y}^T)$.

Since $(\alpha_i, \rho_i)$ is undetectable, by Lemma 1, $(\overline{\alpha}_i^T, \overline{\rho}_i^T)$ weakly increases the stage-game payoff to player $i$ in period $t$. Since $(\overline{\alpha}_i^T, \overline{\rho}_i^T)$ and $(\alpha_i^T, \rho_i^T)$ produce the same distribution of $(\widetilde{a}^T, \widehat{y}^T)$, the expected value of $w_i^{T**}$ is the same under these two strategies. Hence $(\overline{\alpha}_i^T, \overline{\rho}_i^T)$ weakly dominates $(\alpha_i^T, \rho_i^T)$.

## E.2 Step 2

We show that in each case we must have

$$v_i^T(\sigma^{T*}; w_i^{T**}, h_i) > v_i^T(\alpha_i^T, \rho_i^T, \sigma_{-i}^{T*}; w_i^{T**}, h_i),$$

where for any contract $w_i^T$ and strategy profile $\sigma^T$,

$$v_i^T\left(\sigma^T;w_i^T,h_i\right)\equiv\frac{1-\delta}{1-\delta^T}E\left[\sum_{t=1}^{T}\delta^{t-1}g_i(a(t))+w_i^T(\widetilde{a}^T,\widehat{y}^T)\middle|\sigma^T,h_i\right]$$

denotes player $i$'s expected payoff conditional on $\sigma^T$ and $h_i$. By (13), for any $\sigma_i^T$,

$$\begin{aligned}&v_i^T(\sigma_i^T,\sigma_{-i}^{T*};w_i^{T**},h_i)=\\&\qquad\frac{1-\delta}{1-\delta^T}\left(V_i(\sigma_i^T;h_i)-E[w_{i,a}^{T*}(\widetilde{a}^{T\prime},\widehat{y}^{T\prime})|\sigma^{T*}]-\kappa_T-E[\phi_i(\widetilde{a}^T,\widehat{y}^T)|\sigma_i^T,\sigma_{-i}^{T*},h_i]\right),\end{aligned}$$

where

$$V_i(\sigma_i^T;h_i)\equiv E\left[\sum_{s=1}^{T}\delta^{s-1}\left(g_i(a(s))+w_i^*(\widetilde{a}(s),\widehat{y}(s))\right)\middle|\sigma_{-i}^{T*},\sigma_i^T,h_i\right].$$

Hence, noting that $\phi_i\geq 0$ and applying Lemma 4, we have

$$\begin{aligned}&v_i^T(\sigma^{T*};w_i^{T**},h_i)-v_i^T\left(\alpha_i^T,\rho_i^T,\sigma_{-i}^{T*};w_i^{T**},h_i\right)\\=&\frac{1-\delta}{1-\delta^T}\left(V_i(\sigma_i^{T*};h_i)-V_i(\alpha_i^T,\rho_i^T;h_i)\right.\\&\qquad\qquad\left.-E[\phi_i(\widetilde{a}^T,\widehat{y}^T)|\sigma^{T*},h_i]+E[\phi_i(\widetilde{a}^T,\widehat{y}^T)|\alpha_i^T,\rho_i^T,\sigma_{-i}^{T*},h_i]\right)\\\geq&\frac{1-\delta}{1-\delta^T}\left(V_i\left(\sigma_i^{T*};h_i\right)-V_i\left(\alpha_i^T,\rho_i^T;h_i\right)-\varepsilon_T\right).\end{aligned}$$

From the assumption that $\delta^{T-1}\Delta_2\geq\varepsilon_T$, it remains to show that

$$V_i\left(\sigma_i^{T*};h_i\right)-V_i\left(\alpha_i^T,\rho_i^T;h_i\right)>\delta^{T-1}\Delta_2.$$

First consider Case (i). Since $\alpha_i$ is not equivalent to $\alpha_i^*$, by definition, there exists $(\widetilde{a}_{-i},\widetilde{a}_i)$ in the support of $\eta$ and $y_i'$ and $y_i''$ with $p(y_i'|\widetilde{a}_{-i},\alpha_i(\widetilde{a}_i))>0$ and $p(y_i''|\widetilde{a}_{-i},\alpha_i(\widetilde{a}_i))>0$ such that $\rho_i(\widetilde{a}_i,\alpha_i(\widetilde{a}_i),y_i')=\rho_i(\widetilde{a}_i,\alpha_i(\widetilde{a}_i),y_i'')$. By Assumption 1, $p(\cdot|\widetilde{a}_{-i},\alpha_i(\widetilde{a}_i),y_i')\neq p(\cdot|\widetilde{a}_{-i},\alpha_i(\widetilde{a}_i),y_i'')$. Hence $\alpha_i(\widetilde{a}_i)$ is strictly more informative than $\widetilde{a}_i$ at $\widetilde{a}_i$. Since $\pi^{\alpha_i,\rho_i}=\mu$, by (35),

$$v_i\left(\sigma^*;w_i^*\right)-v_i\left(\alpha_i,\rho_i,\sigma_{-i}^*;w_i^*\right)>\Delta_1.$$

If $\rho_i'$ is a reporting strategy such that $(\alpha_i,\rho_i')$ is detectable, then by (34),

$$v_i\left(\sigma^*;w_i^*\right)-v_i\left(\alpha_i,\rho_i',\sigma_{-i}^*;w_i^*\right)>\Delta_0.$$

Since stage-game payoff plus transfer is maximized by $\sigma_i^{T*}$ in each period $s \neq t$,

$$
\begin{aligned}
& V_i\left(\sigma_i^{T*};h_i\right) - V_i\left(\alpha_i^T, \rho_i^T; h_i\right) \\
\geq & \delta^{t-1}\left(E[g_i(a(t)) + w_i^*(\widetilde{a}(t),\widehat{y}(t))|\sigma^{T*},h_i] - E[g_i(a(t)) + w_i^*(\widetilde{a}(t),\widehat{y}(t))|\sigma_{-i}^{T*}, \alpha_i^T, \rho_i^T, h_i]\right) \\
\geq & \delta^{t-1}\min_{\rho_i'}\left(v_i\left(\sigma^*; w_i^*\right) - v_i\left(\alpha_i, \rho_i', \sigma_{-i}^*; w_i^*\right)\right) \\
> & \delta^{t-1}\min\{\Delta_0, \Delta_1\} = \delta^{t-1}\Delta_2.
\end{aligned}
$$

In Case (ii), since $(\alpha_i, \rho_i)$ is detectable for all $\rho_i$,

$$
\begin{aligned}
& V_i\left(\sigma_i^{T*};h_i\right) - V_i\left(\alpha_i^T, \rho_i^T; h_i\right) \\
\geq & \delta^{t-1}\left(E[g_i(a(t)) + w_i^*(\widetilde{a}(t),\widehat{y}(t))|\sigma^{T*},h_i] - E[g_i(a(t)) + w_i^*(\widetilde{a}(t),\widehat{y}(t))|\sigma_{-i}^{T*}, \alpha_i^T, \rho_i^T, h_i]\right) \\
\geq & \delta^{t-1}\min_{\rho_i}\left(v_i\left(\sigma^*; w_i^*\right) - v_i\left(\alpha_i, \rho_i, \sigma_{-i}^*; w_i^*\right)\right) \\
> & \delta^{t-1}\Delta_0.
\end{aligned}
$$

In Case (iii), let $\widehat{y}^T = \rho_i^T(h_i)$ and let $t$ be a period such that $\widehat{y}_i(t) \neq y_i(t)$. Then $(\alpha_i^*, \rho_i)$ is detectable for any $\rho_i$ such that $\rho_i(\widetilde{a}_i(t), \widetilde{a}_i(t), y_i(t)) = \widehat{y}(t)$. Hence,

$$
\begin{aligned}
& V_i\left(\sigma_i^{T*};h_i\right) - V_i\left(\alpha_i^{T*}, \rho_i^T; h_i\right) \\
\geq & \delta^{t-1}\left(E[g_i(a(t)) + w_i^*(\widetilde{a}(t),\widehat{y}(t))|\sigma^{T*},h_i] - E[g_i(a(t)) + w_i^*(\widetilde{a}(t),\widehat{y}(t))|\sigma_{-i}^{T*}, \alpha_i^{T*}, \rho_i^T, h_i]\right) \\
\geq & \delta^{t-1}\min_{\rho_i:\,\rho_i(\widetilde{a}_i(t),\widetilde{a}_i(t),y_i(t))=\widehat{y}(t)}\left(v_i\left(\sigma^*; w_i^*\right) - v_i\left(\alpha_i^*, \rho_i, \sigma_{-i}^*; w_i^*\right)\right) \\
> & \delta^{t-1}\Delta_0.
\end{aligned}
$$

# F Proof of Theorem 3B

Let

$$
\overline{L} \equiv \begin{cases} \sup_{\sigma \in Q(\eta)\setminus K(\eta)} \frac{\sum_{i=1}^n d(\sigma_i)}{l(\sigma_1)-1}, & \text{if } Q(\eta)\setminus K(\eta) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}
$$

To prove Theorem 3B, it remains to show that

$$
L^*(\eta) \leq \overline{L}. \tag{36}
$$

By definition, a contract $w$ enforces $\eta$ with $L(\eta,w) \leq \overline{L}$ if and only if

$$\sum_{(\widetilde{a},\widehat{y})\in A\times Y} [\pi^{\alpha_i,\rho_i}(\widetilde{a},\widehat{y}) - \mu(\widetilde{a},\widehat{y})]\, w_i(\widetilde{a},\widehat{y}) \quad \leq \quad -d_i(\alpha_i,\rho_i) \quad \text{for all } (\alpha_i,\rho_i,i); \tag{37}$$

$$\sum_{i=1}^{n} \sum_{(\widetilde{a},\widehat{y})\in A\times Y} [-\mu(\widetilde{a},\widehat{y}) + \mu(\widetilde{a},\widehat{y}|\omega)]\, w_i(\widetilde{a},\widehat{y}) \quad \leq \quad \overline{L} \quad \text{for all } \omega \in P. \tag{38}$$

By the theorem of alternatives (see, e.g., Proposition 5.1.2 of Bertsekas, 2009), (37) and (38) does not have a solution in $w$ if and only if there exist $\{\lambda_i(\alpha_i,\rho_i) \geq 0 \mid (\alpha_i,\rho_i,i)\}$ and $\{\nu(\omega) \geq 0 \mid \omega \in P\}$ such that

$$\sum_{(\alpha_i,\rho_i)} \lambda_i(\alpha_i,\rho_i)\,[\pi^{\alpha_i,\rho_i}(\cdot) - \mu(\cdot)] + \sum_{\omega\in P} \nu(\omega)\,[-\mu(\cdot) + \mu(\cdot|\omega)] \quad = \quad 0 \text{ for each } i \tag{39}$$

$$-\sum_{i=1}^{n} \sum_{(\alpha_i,\rho_i)} \lambda_i(\alpha_i,\rho_i) d_i(\alpha_i,\rho_i) + \sum_{\omega\in P} \nu(\omega)\overline{L} \quad < \quad 0. \tag{40}$$

Suppose (39) and (40) hold. From (40), $\overline{\lambda} \equiv \max_{i\in N} \sum_{(\alpha_i,\rho_i)} \lambda_i(\alpha_i,\rho_i) > 0$. We can, therefore, define a mixed strategy $\sigma_i$ for each player $i$ such that, for all $(\alpha_i,\rho_i,i)$,

$$\sigma_i(\alpha_i,\rho_i) \equiv \begin{cases} \frac{\lambda_i(\alpha_i,\rho_i)}{\overline{\lambda}}, & \text{if } (\alpha_i,\rho_i) \neq (\alpha_i^*,\rho_i^*); \\ 1 - \sum_{(\alpha_i,\rho_i)\neq(\alpha_i^*,\rho_i^*)} \frac{\lambda_i(\alpha_i,\rho_i)}{\overline{\lambda}}, & \text{otherwise.} \end{cases}$$

Using the definition of $\sigma_i$, we can rewrite (39) and (40) as

$$\overline{\lambda}\,[\pi^{\sigma_i}(\cdot) - \mu(\cdot)] + \sum_{\omega\in P} \nu(\omega)\,[-\mu(\cdot) + \mu(\cdot|\omega)] \quad = \quad 0 \text{ for each } i \tag{41}$$

$$-\sum_{i=1}^{n} \overline{\lambda} d_i(\sigma_i) + \sum_{\omega\in P} \nu(\omega)\overline{L} \quad < \quad 0. \tag{42}$$

Fix a contract $w$. Multiplying each (41) by $w_i(\cdot)$, then summing over all $i$ and all $(\widetilde{a},\widehat{y}) \in A\times Y$, and adding (42), we have

$$\sum_{i=1}^{n} \overline{\lambda} \left( \sum_{(\widetilde{a},\widehat{y})\in A\times Y} [\pi^{\sigma_i}(\widetilde{a},\widehat{y}) - \mu(\widetilde{a},\widehat{y})]\, w_i(\widetilde{a},\widehat{y}) - d_i(\sigma_i) \right)$$
$$+ \sum_{\omega\in P} \nu(\omega) \left( \sum_{(\widetilde{a},\widehat{y})\in A\times Y} [-\mu(\widetilde{a},\widehat{y}) + \mu(\widetilde{a},\widehat{y}|\omega)]\, w_i(\widetilde{a},\widehat{y}) - \overline{L} \right) < 0.$$

Hence, either $L(\eta,w) > \overline{L}$, or

$$\sum_{(\widetilde{a},\widehat{y})\in A\times Y} [\pi^{\sigma_i}(\widetilde{a},\widehat{y}) - \mu(\widetilde{a},\widehat{y})]\, w_i(\widetilde{a},\widehat{y}) - d_i(\sigma_i) < 0$$

for some player $i$. This means that if $\eta$ cannot be enforced by any $w$ with $L(\eta,w) \leq \overline{L}$, then there must exist $\sigma$ such that, for any $w$ with $L(\eta,w) \leq \overline{L}$,

$$v_i(\sigma_i,\sigma^*_{-i};w_i) > v_i(\sigma^*;w_i)$$

for some player $i$.

We prove (36) by showing that for all $\sigma \in \Sigma$, there exists a contract $w$ such that $v_i(\sigma_i,\sigma^*_{-i};w_i) - v_i(\sigma^*;w_i) \leq 0$ for all $i$ and $L(\eta,w) \leq \overline{L}$. By Theorem 4(i) of Rahman and Obara (2010), if $\sigma$ is either unprofitable or attributable, then it can be deterred by a contract with total transfer summing to zero. It remains to consider $\sigma$ such that $\sigma_1 = \cdots = \sigma_n$ and $\sum_{i=1}^n d_i(\sigma_i) > 0$. Since $\eta$ is enforceable, $\pi^{\sigma_i} \neq \mu$.

Case 1. If there exists $(\widetilde{a},\widehat{y})$ such that $\pi^{\sigma_i}(\widetilde{a},\widehat{y}) > 0$ and $\mu(\widetilde{a},\widehat{y}) = 0$, then $\sigma$ can be deterred by a contract $w$ that punishes every player severely whenever $(\widetilde{a},\widehat{y})$ occurs. Clearly, $L(\eta,w) = 0$.

Case 2. Suppose that $\pi^{\sigma_i}(\cdot|\omega) \neq \mu(\cdot|\omega)$ for some $\omega \in P$. Then $\pi^{\sigma_i}(\widetilde{a},\widehat{y}|\omega) > \mu(\widetilde{a},\widehat{y}|\omega)$ for some $(\widetilde{a},\widehat{y}) \in \omega$. We define a contract $w$ by letting, for each $i$,

$$w_i(\widetilde{a}',\widehat{y}') = \begin{cases} -c, & \text{if } (\widetilde{a}',\widehat{y}') = (\widetilde{a},\widehat{y}); \\ -c\cdot\mu(\widetilde{a},\widehat{y}|\omega), & \text{if } (\widetilde{a}',\widehat{y}') \notin \omega; \\ 0, & \text{otherwise.} \end{cases}$$

Then $E[w_i(\widetilde{a}',\widehat{y}')|\sigma^*,\omega'] = -c\cdot\mu(\widetilde{a},\widehat{y}|\omega)$ for all $\omega' \in P$. Hence $L(\eta,w) = 0$. Moreover,

$$v_i(\sigma_i,\sigma^*_{-i};w_i) - v_i(\sigma^*;w_i) = -c\cdot(\pi^{\sigma_i}(\widetilde{a},\widehat{y}|\omega) - \mu(\widetilde{a},\widehat{y}|\omega))\pi^{\sigma_i}(\omega) + d(\sigma_i) \leq 0,$$

when $c$ is large enough.

Case 3. Suppose that $\sigma \in Q(\eta)\setminus K(\eta)$. Let $\omega$ solve $\max_{\omega'\in P}\frac{\pi^{\sigma_i}(\omega')}{\mu(\omega')}$. We define a contract $w$ by letting, for each $i$,

$$w_i(\widetilde{a},\widehat{y}) = \begin{cases} -\frac{d(\sigma_i)}{\pi^{\sigma_i}(\omega)-\mu(\omega)}, & \text{if } (\widetilde{a},\widehat{y}) \in \omega; \\ 0, & \text{otherwise.} \end{cases}$$

Then $L(\eta,w)=\frac{\sum_{i=1}^{n}d(\sigma_i)}{\pi^{\sigma_1}(\omega)-\mu(\omega)}\mu(\omega)=\frac{\sum_{i=1}^{n}d(\sigma_i)}{l(\sigma_1)-1}$ and

$$v_i(\sigma_i,\sigma^*_{-i};w_i)-v_i(\sigma^*;w_i)=-\frac{d(\sigma_i)}{\pi^{\sigma_i}(\omega)-\mu(\omega)}\cdot(\pi^{\sigma_i}(\omega)-\mu(\omega))+d(\sigma_i)=0.$$

# G Proof of Theorem 4

We prove the theorem in four steps.

## G.1 Step 1. Defining $\overline{\eta}$.

For each player $i$ and each $(a^*_{-i},y^*_{-i})\in A_{-i}\times Y_{-i}$ such that $\mu(a^*_{-i},a_i,y^*_{-i},y_i)>0$ for some $(a_i,y_i)$, we fix an action $f_i(a^*_{-i},y^*_{-i})\in A_i$ that satisfies the following conditions.

(i) We have $p(y^*_{-i}|a^*_{-i},a_i)\le p(y^*_{-i}|a^*_{-i},f_i(a^*_{-i},y^*_{-i}))$ for all $a_i$.

(ii) If $p(y^*_{-i}|a^*_{-i},a_i)=p(y^*_{-i}|a^*_{-i},f_i(a^*_{-i},y^*_{-i}))$, then $g_i(a^*_{-i},a_i)\le g_i(a^*_{-i},f_i(a^*_{-i},y^*_{-i}))$.

(iii) The action $f_i(a^*_{-i},y^*_{-i})$ satisfies the no-free-information condition when the other players are choosing $a^*_{-i}$.

Let $B'_i$ be the set of actions in $A_i$ that maximizes the marginal probability $p(y^*_{-i}|a^*_{-i},a_i)$. Let $B_i=\arg\max_{a_i\in B'_i}g_i(a^*_{-i},a_i)$. Any $a_i\in B_i$ satisfies Conditions (i) and (ii). We show that there exists $f_i(a^*_{-i},y^*_{-i})\in B_i$ that satisfies Condition (iii).

Without loss of generality, assume that each $a_i\in B_i$ is associated with a different distribution of posterior beliefs. For each $a_i\in B_i$, let

$$\Phi(a_i,y_i;\widehat{a}_i,\widehat{y}_i)\equiv\sum_{y_{-i}\in Y_{-i}}p(y_{-i}|a^*_{-i},a_i,y_i)\varphi(y_{-i}|a^*_{-i},\widehat{a}_i,\widehat{y}_i),$$

where

$$\varphi(y_{-i}|a^*_{-i},\widehat{a}_i,\widehat{y}_i)\equiv 2p(y_{-i}|a^*_{-i},\widehat{a}_i,\widehat{y}_i)-\sum_{y'_{-i}\in Y_{-i}}\left(p(y'_{-i}|a^*_{-i},\widehat{a}_i,\widehat{y}_i)\right)^2$$

is a scoring rule. Hence

$$\Phi(a_i,y_i;a_i,y_i)\ge\Phi(a_i,y_i;\widehat{a}_i,\widehat{y}_i),$$

with the inequality strict whenever $p(\cdot|a^*_{-i},a_i,y_i) \neq p(\cdot|a^*_{-i},\widehat{a}_i,\widehat{y}_i)$.

Let $\Phi^*(a_i) \equiv \sum_{y_i \in Y_i} p(y_i|a^*_{-i},a_i)\Phi(a_i,y_i;a_i,y_i)$. We claim that any $a^*_i$ that maximizes $\Phi^*(a_i)$ among those $a_i$ in $B_i$ satisfies the no-free-information condition. Suppose, by way of contradiction, that the claim is wrong. By construction, if $\gamma_i$ is strictly more informative than $a^*_i$, then its support is contained in $B'_i$. If its support contains some $a_i \notin B_i$, then the expected stage-game payoff under $\gamma_i$ is strictly lower than that under $a^*_i$, against $a^*_{-i}$. Hence, if the claim is wrong, then there exists $\gamma_i$ with support contained in $B_i$ and $\lambda_{(a_i,y_i)}(\cdot) \in \Delta(Y_i)$ for each $(a_i,y_i) \in A_i \times Y_i$ satisfying the following conditions.

(iv) For all $y_{-i} \in Y_{-i}$ and all $y'_i \in Y_i$,

$$\sum_{(a_i,y_i)\in A_i\times Y_i} \gamma_i(a_i)\lambda_{(a_i,y_i)}(y'_i)p(y_i|a^*_{-i},a_i)p(y_{-i}|a^*_{-i},a_i,y_i) = p(y_{-i},y'_i|a^*_{-i},a^*_i).$$

(v) There exist $(a_i,y_i)$ and $y''_i$ such that $\gamma_i(a_i) > 0$, $p(y_i|a^*_{-i},a_i) > 0$, $\lambda_{(a_i,y_i)}(y''_i) > 0$, and $p(\cdot|a^*_{-i},a_i,y_i) \neq p(\cdot|a^*_{-i},a^*_i,y''_i)$.

It follows from Condition (iv) that

$$\sum_{a_i\in A_i} \gamma_i(a_i) \sum_{y_i\in Y_i} p(y_i|a^*_{-i},a_i) \sum_{y'_i\in Y_i} \lambda_{(a_i,y_i)}(y'_i)\Phi(a_i,y_i;a^*_i,y'_i) = \Phi^*(a^*_i).$$

However,

$$\begin{aligned}\sum_{a_i\in A_i} \gamma_i(a_i)\Phi^*(a_i) &= \sum_{a_i\in A_i} \gamma_i(a_i) \sum_{y_i\in Y_i} p(y_i|a^*_{-i},a_i)\Phi(a_i,y_i;a_i,y_i)\\ &> \sum_{a_i\in A_i} \gamma_i(a_i) \sum_{y_i\in Y_i} p(y_i|a^*_{-i},a_i) \sum_{y'_i\in Y_i} \lambda_{(a_i,y_i)}(y'_i)\Phi(a_i,y_i;a^*_i,y'_i) = \Phi^*(a^*_i).\end{aligned}$$

The strict inequality follows from Condition (v). This contradicts the supposition that $a^*_i$ maximizes $\Phi^*(a_i)$.

Let $k$ be the cardinality of the set

$$\overline{A} \equiv \{(a^*_{-i}, f_i(a^*_{-i},y^*_{-i}))|(a^*_{-i},y^*_{-i}) \in A_{-i}\times Y_{-i}, \mu(a^*_{-i},a_i,y^*_{-i},y_i) > 0 \text{ for some } (a_i,y_i), i \in N\}.$$

For any action profile $\widetilde{a} \in A$, we let $\phi_{\widetilde{a}} \in \Delta(A)$ denote the distribution that assigns probability 1 to $\widetilde{a}$. That is, $\phi_{\widetilde{a}}(\widetilde{a}) = 1$ and $\phi_{\widetilde{a}}(\widetilde{a}') = 0$ for all $\widetilde{a}' \neq \widetilde{a}$. Define

$$\overline{\eta} \equiv (1-\varepsilon)\eta + \frac{\varepsilon}{k}\sum_{\widetilde{a}\in\overline{A}} \phi_{\widetilde{a}}.$$

By definition, for each $\widetilde{a}' \in A$,

$$|\eta(\widetilde{a}') - \overline{\eta}(\widetilde{a}')| \leq \frac{\varepsilon}{k} \sum_{\widetilde{a} \in \overline{A}} |\eta(\widetilde{a}') - \phi_{\widetilde{a}}(\widetilde{a}')| \leq \varepsilon.$$

Let $\overline{\mu}$ be the joint distribution of recommendation and signal under $\overline{\eta}$, i.e. $\overline{\mu}(\widetilde{a},y) = p(y|\widetilde{a})\overline{\eta}(\widetilde{a})$ for all $(\widetilde{a},y)$. Let $\overline{P}$ be the meet of the information partitions of the players under $\overline{\mu}$ and let $\overline{\varpi}$ denote a generic element of $\overline{P}$. We use $d(\eta,\sigma_i)$ and $d(\overline{\eta},\sigma_i)$ to denote player $i$'s gains in the stage-game payoff from using the strategy $\sigma_i$, when the other players use $\sigma^*_{-i}$ and the underlying action distributions are $\eta$ and $\overline{\eta}$, respectively.

We first establish a claim. Let $S'_i$ denote the set of pure stage-game strategies $(\alpha_i,\rho_i)$ such that either $\alpha_i(\widetilde{a}_i) \neq \widetilde{a}_i$ for some $\widetilde{a}_i$ in the support of $\eta_i$ or $\pi^{\alpha_i,\rho_i} \neq \mu$.

**Claim 2.** *There exists $\overline{\varepsilon} > 0$ such that for any $\varepsilon < \overline{\varepsilon}$, any player $i \in N$, and any $\sigma_i$ such that $\pi^{\sigma_i} = \mu$ and its support contains some $(\alpha_i,\rho_i) \in S'_i$, we have*

$$\sum_{(\alpha_i,\rho_i)} \sigma_i(\alpha_i,\rho_i) \sum_{\widetilde{a}_i \in \text{supp}(\eta_i)} \sum_{\widetilde{a}_{-i} \in A_{-i}} (g_i(\widetilde{a}_{-i},\alpha_i(\widetilde{a}_i)) - g_i(\widetilde{a}))\overline{\eta}(\widetilde{a}) < 0. \tag{43}$$

Equation (43) says that $\sigma_i$ is strictly unprofitable when player $i$ is recommended to choose actions in the support of $\eta_i$ while other players' recommendations are drawn according to $\overline{\eta}$.

To prove the claim, note that $\eta$ is strictly enforceable, hence there exists a contract $w$ such that

$$v_i(\sigma^*;w_i) - v_i(\sigma^*_{-i},\alpha_i,\rho_i;w_i) > 0$$

for all $i$ and all $(\alpha_i,\rho_i) \in S'_i$. Since the set of pure strategies $(\alpha_i,\rho_i)$ is finite, there exists $c_0 > 0$ such that this inequality can be strengthened to

$$v_i(\sigma^*;w_i) - v_i(\sigma^*_{-i},\alpha_i,\rho_i;w_i) > c_0.$$

It follows that for all $\sigma'_i$ such that $\pi^{\sigma'_i} = \mu$ and the support of $\sigma'_i$ is contained in $S'_i$,

$$\begin{aligned} d(\eta,\sigma'_i) =& v_i(\sigma^*_{-i},\sigma'_i;w_i) - v_i(\sigma^*;w_i) \\ &= \sum_{(\alpha_i,\rho_i)} \sigma'_i(\alpha_i,\rho_i)(v_i(\sigma^*_{-i},\alpha_i,\rho_i;w_i) - v_i(\sigma^*;w_i)) < -c_0. \end{aligned}$$

Hence, there exists $\overline{\varepsilon} > 0$, uniform for all $\sigma_i' \in \Delta(S_i')$ such that $\pi^{\sigma_i'} = \mu$ and all players, such that for any $\varepsilon < \overline{\varepsilon}$, we have

$$\sum_{(\alpha_i,\rho_i)} \sigma_i'(\alpha_i,\rho_i) \sum_{\widetilde{a}_i \in \mathrm{supp}(\eta_i)} \sum_{\widetilde{a}_{-i} \in A_{-i}} (g_i(\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) - g_i(\widetilde{a}))\overline{\eta}(\widetilde{a}) < 0.$$

Now for any $\sigma_i$ such that $\pi^{\sigma_i} = \mu$ and its support contains some $(\alpha_i,\rho_i) \in S_i'$, we can write $\sigma_i = x\sigma_i' + (1-x)\sigma_i''$ for some $x \in (0,1]$ such that the support of $\sigma_i'$ is contained in $S_i'$ and the support of $\sigma_i''$ is outside $S_i'$. Since $\pi^{\alpha_i,\rho_i} = \mu$ for each $(\alpha_i,\rho_i) \notin S_i'$, we have $\pi^{\sigma_i'} = \mu$. By the conclusion in the last paragraph and the fact that $(\alpha_i,\rho_i) \notin S_i'$ implies $\alpha_i(\widetilde{a}_i) = \widetilde{a}_i$ for all $\widetilde{a}_i$ in the support of $\eta_i$, we have

$$\begin{aligned}\sum_{(\alpha_i,\rho_i)} \sigma_i(\alpha_i,\rho_i) \sum_{\widetilde{a}_i \in \mathrm{supp}(\eta_i)} \sum_{\widetilde{a}_{-i} \in A_{-i}} (g_i(\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) - g_i(\widetilde{a}))\overline{\eta}(\widetilde{a}) = \\ x \sum_{(\alpha_i,\rho_i)} \sigma_i'(\alpha_i,\rho_i) \sum_{\widetilde{a}_i \in \mathrm{supp}(\eta_i)} \sum_{\widetilde{a}_{-i} \in A_{-i}} (g_i(\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) - g_i(\widetilde{a}))\overline{\eta}(\widetilde{a}) < 0.\end{aligned}$$

This proves Claim 2.

## G.2 Step 2. Showing that $\overline{\eta}$ is enforceable.

Suppose that $\overline{\pi}^{\sigma_i} = \overline{\mu}$. Then $\pi^{\sigma_i} = \mu$. If the support of $\sigma_i$ contains some $(\alpha_i,\rho_i) \in S_i'$, by Claim 2 we have

$$\sum_{(\alpha_i,\rho_i)} \sigma_i(\alpha_i,\rho_i) \sum_{\widetilde{a}_i \in \mathrm{supp}(\eta_i)} \sum_{\widetilde{a}_{-i} \in A_{-i}} (g_i(\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) - g_i(\widetilde{a}))\overline{\eta}(\widetilde{a}) < 0. \tag{44}$$

If $(\alpha_i,\rho_i) \notin S_i'$ for each $(\alpha_i,\rho_i)$ in the support of $\sigma_i$, since $(\alpha_i,\rho_i) \notin S_i'$ implies that $\alpha_i(\widetilde{a}_i) = \widetilde{a}_i$ for all $\widetilde{a}_i$ in the support of $\eta_i$, then

$$\sum_{(\alpha_i,\rho_i)} \sigma_i(\alpha_i,\rho_i) \sum_{\widetilde{a}_i \in \mathrm{supp}(\eta_i)} \sum_{\widetilde{a}_{-i} \in A_{-i}} (g_i(\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) - g_i(\widetilde{a}))\overline{\eta}(\widetilde{a}) = 0. \tag{45}$$

For each $\widetilde{a}_i$ that equals $f_i(\widetilde{a}_{-i}, y_{-i}^*)$ for some $y_{-i}^*$, by Condition (ii) we have, for any $\alpha_i$ in the support of $\sigma_i$,

$$g_i(\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) \leq g_i(\widetilde{a}). \tag{46}$$

Combining (44), (45) and (46), we have

$$d(\overline{\eta}, \sigma_i) = \sum_{(\alpha_i,\rho_i)} \sigma_i(\alpha_i,\rho_i) \sum_{\widetilde{a} \in A} (g_i(\widetilde{a}_{-i}, \alpha_i(\widetilde{a}_i)) - g_i(\widetilde{a}))\overline{\eta}(\widetilde{a}) \leq 0.$$

## G.3 Step 3. Showing that $\overline{\eta}$ satisfies the no-free-information condition.

Suppose that $\widetilde{a}_i \in \mathrm{supp}(\eta_i)$ and $\gamma_i$ is strictly more informative than $\widetilde{a}_i$ at $\widetilde{a}_i$ under $\overline{\eta}$. Then there is a mixed strategy $\sigma_i$ such that $\sum_{(\alpha_i,\rho_i):\alpha_i(\widetilde{a}_i)=a_i} \sigma_i(\alpha_i,\rho_i) = \gamma_i(a_i)$, $\alpha_i(\widetilde{a}'_i) = \widetilde{a}'_i$ for all $\widetilde{a}'_i \neq \widetilde{a}_i$, and $\overline{\pi}^{\sigma_i} = \overline{\mu}$. By Claim 2,

$$\sum_{a_i \in A_i} \gamma_i(a_i) \sum_{\widetilde{a}_{-i} \in A_{-i}} (g(a_i,\widetilde{a}_{-i}) - g(\widetilde{a}))\overline{\eta}(\widetilde{a}) =$$
$$\sum_{(\alpha_i,\rho_i)} \sigma_i(\alpha_i,\rho_i) \sum_{\widetilde{a}'_i \in \mathrm{supp}(\eta_i)} \sum_{\widetilde{a}_{-i} \in A_{-i}} (g_i(\widetilde{a}_{-i},\alpha_i(\widetilde{a}'_i)) - g_i(\widetilde{a}_{-i},\widetilde{a}'_i))\overline{\eta}(\widetilde{a}_{-i},\widetilde{a}'_i) < 0.$$

Suppose that $\widetilde{a}_i = f_i(a^*_{-i},y^*_{-i})$ for some $(a^*_{-i},y^*_{-i})$. By Condition (iii), $\widetilde{a}_i$ satisfies the no-free-information condition.

## G.4 Step 4. Showing that $L^*(\overline{\eta}) = 0$.

We show that any strategy $\sigma_i \neq \sigma^*_i$ satisfies either $\overline{\pi}^{\sigma_i} = \overline{\mu}$, or there exists $(\widetilde{a},\widehat{y})$ such that $\overline{\pi}^{\sigma_i}(\widetilde{a},\widehat{y}) > 0$ but $\overline{\mu}(\widetilde{a},\widehat{y}) = 0$, or $\overline{\pi}^{\sigma_i}(\,\cdot\,|\overline{\omega}) \neq \overline{\mu}(\,\cdot\,|\overline{\omega})$ for some $\overline{\omega}$. Hence, by Theorem 3B, $L^*(\overline{\eta}) = 0$.

Suppose that $\overline{\pi}^{\sigma_i} \neq \overline{\mu}$ and for all $(\widetilde{a},\widehat{y})$ such that $\overline{\pi}^{\sigma_i}(\widetilde{a},\widehat{y}) > 0$, we have $\overline{\mu}(\widetilde{a},\widehat{y}) > 0$. Then there exists $(\widetilde{a},\widehat{y})$ such that

$$\overline{\pi}^{\sigma_i}(\widetilde{a},\widehat{y}) > \overline{\mu}(\widetilde{a},\widehat{y}) > 0. \tag{47}$$

Let $\overline{\omega}$ be the event in $\overline{P}$ that contains $(\widetilde{a},\widehat{y})$. By Condition (i), we have

$$\sum_{\widehat{y}'_i \in Y_i} \overline{\pi}^{\sigma_i}(\widetilde{a}_{-i}, f_i(\widetilde{a}_{-i},\widehat{y}_{-i}),\widehat{y}_{-i},\widehat{y}'_i) \leq \sum_{\widehat{y}'_i \in Y_i} \overline{\mu}(\widetilde{a}_{-i}, f_i(\widetilde{a}_{-i},\widehat{y}_{-i}),\widehat{y}_{-i},\widehat{y}'_i). \tag{48}$$

Combining (47) and (48) yields

$$\frac{\overline{\pi}^{\sigma_i}(\widetilde{a},\widehat{y})}{\sum_{\widehat{y}'_i \in Y_i} \overline{\pi}^{\sigma_i}(\widetilde{a}_{-i}, f_i(\widetilde{a}_{-i},\widehat{y}_{-i}),\widehat{y}_{-i},\widehat{y}'_i)} > \frac{\overline{\mu}(\widetilde{a},\widehat{y})}{\sum_{\widehat{y}'_i \in Y_i} \overline{\mu}(\widetilde{a}_{-i}, f_i(\widetilde{a}_{-i},\widehat{y}_{-i}),\widehat{y}_{-i},\widehat{y}'_i)}.$$

Dividing both the numerator and the denominator of the left-hand side by $\overline{\pi}^{\sigma_i}(\overline{\omega})$ and those of the right-hand side by $\overline{\mu}(\overline{\omega})$, we have

$$\frac{\overline{\pi}^{\sigma_i}(\widetilde{a},\widehat{y}|\overline{\omega})}{\sum_{\widehat{y}'_i \in Y_i} \overline{\pi}^{\sigma_i}(\widetilde{a}_{-i}, f_i(\widetilde{a}_{-i},\widehat{y}_{-i}),\widehat{y}_{-i},\widehat{y}'_i|\overline{\omega})} > \frac{\overline{\mu}(\widetilde{a},\widehat{y}|\overline{\omega})}{\sum_{\widehat{y}'_i \in Y_i} \overline{\mu}(\widetilde{a}_{-i}, f_i(\widetilde{a}_{-i},\widehat{y}_{-i}),\widehat{y}_{-i},\widehat{y}'_i|\overline{\omega})}.$$

That is, $\overline{\pi}^{\sigma_i}(\,\cdot\,|\overline{\omega}) \neq \overline{\mu}(\,\cdot\,|\overline{\omega})$.

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